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Consensus in topologically interacting swarms under communication constraints and time-delays

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Abstract The emergence of collective decision in swarming systems underscores the central role played by information transmission. Using network-control- and information-theoretic elements applied to a group of topologically interacting agents seeking consensus under switching topologies, the effects of constraints in the information capacity of the communication channel are investigated. This particular system requires us to contend with constantly reconfigurable and spatially embedded interaction networks. We find a sufficient condition on the information data rate guaranteeing the stability of the consensus process in the noiseless case. This result highlights the profound connection with the topological structure of the underlying interaction network, thus having far-reaching implications in the nascent field of swarm robotics. Furthermore, we analyze the more complex case of combined effect of noise and limited data rate. We find that the consensus process is degraded when decreasing the data rate. Moreover, the relationship between critical noise and data rate is found to be in good agreement with information-theoretic predictions. Lastly, we prove that with not-too-large time-delays, our system of topologically interacting agents is stable, provided the underlying interaction network is strongly connected. Using Lyapunov techniques, the maximum allowed time-delay is determined in terms of linear-matrix inequalities.

Keywords Swarm · Topological interaction · Consensus dynamics · Communication constraints

1 Introduction

Information is a crucial currency in swarming systems. This has important consequences when considering collective behaviors of interacting agents [5]. Information exchanges are paramount to the seamless execution of swarming behaviors such as fish schooling, birds flocking, amoebae aggregating, locusts marching or more generally agents swarming [14,31]. In practice, multiagent systems evolve in dynamic environments, therefore encountering unexpected changes in their surroundings, and their effective operation critically hinges on their prompt collective response in adapting to evolving circumstances. The problem of swarm stability with high-order linear time-invariant (LTI) systems and in the absence of communication constraints has been previously studied [7]. Still with high-order LTI swarming systems, a novel method for clustering has recently been introduced, which is based on quasi-consensus in dynamical systems, yet still in the absence of any communication constraints [6].

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Swarms of mobile robots evolving in unknown environments, however, are bound to face unpredictable obstacles or hurdles, and that in the presence of inter-agent connections that may change dynamically, e.g., due to limited communication range. It is now widely believed that the benefits of swarming in natural systems are directly related to their enhanced adaptivity to dynamic environments [5,18]. Over the past decade, collective information transmission has been recognized to be central to the surprising responsiveness of swarms [10,18]. There is mounting evidence that swarm intelligence, in terms of collective response—a.k.a. flexibility, critically depends on ensuring an appropriate flow of behavioral information among agents, both in terms of quantity and accuracy of the informational exchanges.

Networked control systems (NCSs) and multiagent adaptive systems are engineering embodiments of natural swarms. A key problem with these systems is the design of control algorithms achieving specific collective behaviors with reduced or unreliable data exchanges and switching topologies [1,11]. In the past two decades, significant advances have been achieved paving the way to emerging engineering applications such as the control of distributed sensor networks, the coordination of autonomous—air, surface and underwater—vehicles, robotic swarming, etc. [23]. Information and communication constraints are now recognized as being critical to large-scale NCSs, whose performance and effective operation require appropriate and sufficient information exchanges among the different parts constituting the system [1,11]. Over the past decade, some bridges between control theory and information theory have been established, thereby leading to new insights into the interplay between concepts borrowed from both fields and related to NCSs, along with a host of new theoretical results focusing on fundamental trade-offs between information flow constraints and effective collective dynamics [1,11,19–21,23,32,34,35,38].

All transmission of information—in the broadest possible sense—is associated with a communication channel limited by its structure and capacity. The stability and stabilizability of networked control systems have attracted a significant attention owing to their importance for large-scale networked infrastructure systems. A wide range of results about formal conditions—in the form of necessary and/or sufficient conditions—have been reported for: (a) the information data rate [20,21,32,34,35], but also (b) the channel topology [11,19,23,38].

With the advent of large swarms of mobile robots and cooperative sensory networks [27,41], the problem of maintaining consensus or cooperation under dynamic topologies and with limited communication exchanges has become apparent [8]. The successful design of this new class of engineering systems hinges on the triadic relationship between: (i) required minimum information flow, (ii) topology of the interaction network, and (iii) stability or stabilizability of the collective operation. A clear understanding of this intricate triadic relationship is still lacking. The vast majority of studies have been focused on improving our understanding of the effects of changes in the network topology on collective behaviors [15,23,26,31]. These works were primarily motivated by two factors: (a) our limited grasp of some fine details associated with the functional relevance of swarms, and (b) the rapid progress in the field of design of swarm robotics systems. It is worth stressing that the issue of limited information fluxes—often corresponding to reduced information data rates—is not just paramount to the effectiveness of engineering systems, but is also central to the effectiveness of biological and social systems. For instance, scientists have yet to report a functional explanation for the thwarting of the collective dynamics of networked neuronal cells in spinal ganglions following the injection of a drug that lowers the frequency of firing of cells [28].

Although numerous studies have been dedicated to the effects of reduced data rate [20,21,32,34,35], network topology [11,19,23,38], and time-delays [11,23] in multiagent systems, no report has been made about the combined effect of limited communication bandwidth and noise in the communication channel. In addition, no relationship between the consensus reaching dynamics and the minimum required bandwidth—in terms of eigenvalues of the graph Laplacian of the interaction network—can be found. Moreover, an analysis of these particular issues for a swarm of topologically interacting agents is still lacking, as well as the effect of time-delays on its stability. This paper addresses some of these gaps in the literature of multiagent systems dynamics.

Specifically, this paper investigates the above-mentioned triadic relationship associated with limited information flows owing to a reduced capacity of the communication channel in the particular case of collective behaviors originating from topological interactions and for which information flows through a directed and temporally adaptive interaction network. First, using our prior knowledge of such switching adaptive interaction networks and by invoking the min-flow max-cut theorem, we identify and formalize the different possible origins of information flow bottlenecks. We then focus on the problem of ensuring a coherent swarming behavior and establish mathematically a sufficient condition on the information data rate guaranteeing the emergence of a collective response in the noiseless case. This condition highlights the profound connection between information flow and topology of the interaction network and can serve as design principle for novel swarming systems. We also provide the first investigation of the combined effect of limited data rate and limited signal-to-noise ratio on the achievement of an effective swarming behavior—of the consensus reaching type—associated with a switching interaction topology. As the last step, we quantify the impact of communication delay on the consensus stability and derive the maximum allowed time-delay below which the system remains stable.

2 Problem statement

In this section, we first study the information bottlenecks in swarms and then present a prototypical model used in the paper to formally investigate the studied bottlenecks.

2.1 Informational bottlenecks in collective multiagent dynamics

The mechanistic quest initiated with the self-propelled particles model—SPP in the sequel and originally proposed in the seminal paper by Vicsek and collaborators [33]—has recently focused on gaining a better comprehension of the information transfer within swarms, with the ultimate goal of achieving functional predictions about collective behavior. Here, we use a prototypical model of swarming, which is a refinement to the original model by Vicsek et al. [33] albeit based on a topological interaction distance. In these simplified models of swarming, the collective decision-making process is the outcome of an emergent phenomenon that follows localized interactions among agents yielding a global information flow. As a consequence, to investigate the dynamics of these multiagent systems, one needs to identify the information exchanges and the underlying communication channel. Regardless of the actual structure of the communication channel and nature of the information exchange, there is a finite amount of information able to flow. This defines the capacity, which is known to be limited by both noise and bandwidth of the communication channel.

Recently, borrowing concepts from network theory and focusing on the specificities of the interaction network, a novel approach toward the dynamics of multiagent systems has emerged [9,16,29,31]. This new approach paves the way for the development of an integrated approach toward distributed communication in multiagent systems. For instance, this approach can be well understood using the prototypical swarming behavior of predator avoidance in which the detection of an incoming predator triggers a fright response in the form of a swift directional change in a limited set of agents. These evasive maneuvers are triggered by a limited number of informed agents detecting the threat and locally responding to it. These changes in behavior of the informed agents form a signal transmitted through the interaction network (IN). Specifically, the behavioral change propagates between nodes of the network (see schematics diagram in Fig. 1), thereby finding a path that depends on the connectivity of the IN. Our prior works have shown that the connectedness of the IN critically depends on the interaction distance—being a metric one, a topological one, or even a hybrid metric/topological distance [30]—as well as the density of interacting units. Moreover, the IN topology critically affects the consensus reaching process at the core of swarming [31]. Specifically, when considering agents interacting by means of a topological interaction—similarly to flocking starlings, we have established that the IN is a small-world network with a homogeneous degree distribution yielding clustering coefficients of the order of 0.6 [16]. It worth adding that these INs are temporal networks [12], whose dynamics *on* the network drives the network dynamics as a consequence of the spatial embedding of the swarming agents. Hence, the dynamical laws governing the agents' behavior can be used at the system level to develop a MIMO analysis of such NCSs [1].

Beyond the control-theoretic aspects of this problem, a complete analysis of the process of information propagation within a multiagent systems fur-

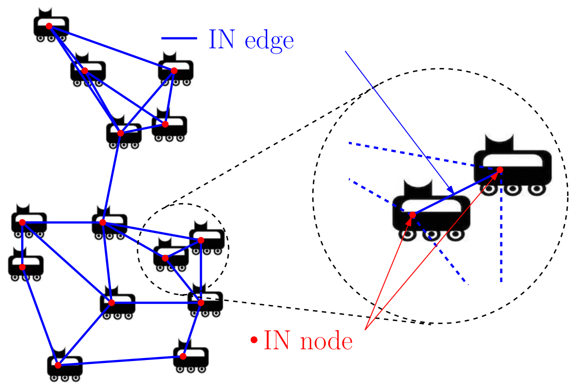


Fig. 1 Schematic of a swarm of networked mobile platforms. The interaction network (IN) is represented with the topologically interacting agents (nodes are shown as red dots) connected by means of network edges (blue links). (Color figure online)

ther requires resorting to information-theoretic concepts. Through the careful integration of concepts belonging to both of these theoretical frameworks, researchers have established a number of fundamental results for NCSs and multiagent systems [1,11,19–21,23,32,34,35,38]. As is well known, the Shannon–Hartley (SH) theorem gives the maximum capacity of any communication channel in terms of its noisiness and bandwidth. Mathematically, this theorem reads as

$$C = B \log_2(1 + \mathcal{S}), \tag{1}$$

where C is the channel capacity, B its bandwidth, and $\mathcal{S}$ the signal-to-noise ratio [17].

In the case of dynamic multiagent systems, one has to consider the information capacities C_e and C_n associated with edges and nodes of the network, respectively. We remark that C_e is due to inter-agent informational signaling through the medium. Correspondingly, C_n is associated with the information capacity of the combined sensory, control, and mobility apparatus internal to each agent (node) which serves as router for the state information (see Fig. 1).

Bottlenecks in the propagation of information through a networked dynamical system can readily be identified using the max-flow min-cut (MFMC) theorem [24] applied to the IN. Specifically, the MFMC states that the maximum flow rate through the IN is obtained when the capacity of edges *or* nodes is minimum. In our particular framework, this important result takes on a particular significance: the process of information propagation within a swarm can be hindered by a reduced flow in the IN edges—it can be the environment for natural swarms or electromagnetic waves for artificial ones, or by the IN node routing capacity—cognitive capabilities of the swarming agents for a natural swarm or data processing capabilities of the units making a swarm robotics system.

The SH theorem (1) therefore reveals that information propagation in a swarm is either limited by B, or by $\mathcal{S}$. By coupling this result to the what has just been said based on the MFMC theorem, one can conclude to the existence of four possible sources of jam in information flow:

(a) reduced $\mathcal{S}_\mathrm{e}$, typically due to the noisiness of the environment in which the agents are evolving;
(b) reduced $\mathcal{S}_\mathrm{n}$, at the swarming unit level;
(c) reduced B_e of the inter-unit transmission medium;
(d) reduced B_n of the data processing unit at the node level.

In what follows, we disregard options (a) and (c) above, as they are mostly related to environmental conditions, while we center our attention on options (b) and (d)—the most probable for engineered multiagent systems. Our ultimate goal is to investigate collapses in information flow originating from reduced levels of B_n, while also being affected by possibly high node-level noises η_n, or equivalently low $\mathcal{S}_\mathrm{n}$.

2.2 Swarming model

We consider a minimalist model for the swarming system, which consists of N locally interacting self-propelled particles (SPPs) [16,31,33], moving at constant speed v_0 through a $\ell \times \ell$ domain having periodic boundaries. Each unit i is fully represented by the state variable θ_i corresponding to its travel direction, and a canonical swarming behavior of the consensus type is examined. Specifically, when individual units of a multiagent system reach an agreement on the value of a certain state variable—in our case the agent's heading θ_i—a consensus is said to have been attained. Such a consensus ensures that units involved in collective information transfer over a given IN have a group-level knowledge of the information required to reach some form of collective action. This is achieved through a consensus algorithm, which imposes identical dynamics on the consensus variable of each vehicle.

Mathematically, consensus is attained when all state variables converge to a single value in the asymptotic sense $\theta_1 = \theta_2 = \cdots = \theta_N$ [23]. When considering collective motion, canonical consensus algorithms simply amount to first-order angular alignment rules generically expressed as

$$\dot{\theta}_i(t) = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \omega_{ij}(\theta_j(t) - \theta_i(t)), \tag{2}$$

$\mathcal{N}_i$ representing the set of neighboring agents of i in the network sense, with $|\mathcal{N}_i|$ the number of neighboring agents, and ω_{ij} the weight of the undirected edge $i - j$ with unit value if i is in direct interaction with j, and 0 otherwise. It is worth noting that more general weights can readily be considered [5].

The dynamics of any swarming system is the outcome of repeated local interactions among units. There is a vast range of possible interaction rules; each rule being fully characterized by an interaction distance and its associated neighborhood (see Fig. 2). The most classical interaction distance is the metric one [33], for which information transfers occur only among agents located at a fixed distance (R in Fig. 2). For simplicity, the radius R is often considered to be identical for all agents. Recent empirical evidence in flocks of starling and human crowds have revealed the existence of metric-free interactions in those systems. Specifically, agents interact topologically with a fixed number of neighbors (see radius r in Fig. 2a, b). Fundamentally, each interaction type is related to a particular shortcoming at the agent level. In the metric case, the value of R can be traced to the range of the sensory suite of the swarming units. In the topological case, the fixed number of interacting agents is often said to be associated with the limited data processing capability of each unit.

Here, given our focus on the effects of limited information capacity, we consider a topological neighborhood for primarily two reasons: (1) it affords all agents with exactly the same communication capabilities, i.e., each agent can establish a fixed number of links with its k-nearest neighbors, and (2) it is more appropriate for agents limited by information-processing capabilities rather than sensory capabilities. However, qualitatively similar results were obtained with the exact same model with metric interactions, which is consistent with the recent proof of a unique universality class in the noise-induced criticality of multiagent SPPs, regardless of the metric or topological nature of interactions [2].

To account for the finiteness of the bandwidth, we consider synchronous information exchanges occurring every

$$T_\mathrm{n} = \frac{1}{2B_\mathrm{n}}, \tag{3}$$

where T_n is the interval of time between changes in a signal transmitted over a given communication channel, a.k.a. unit interval [17]. The agents move synchronously at discrete time steps T_n by a fixed distance $\delta = v_0 T_\mathrm{n}$ upon receiving informational signals from their neighbors as per the linear update rule each agent i

$$\theta_i(t+T_\mathrm{n}) = \theta_i(t) + \frac{T_\mathrm{n}}{k} \sum_{j \in \mathcal{N}_i(t)} \left\{\theta_j(t) - \theta_i(t)\right\} + \eta_\mathrm{n}\xi_i(t), \tag{4}$$

with $\mathcal{N}_i(t)$ representing its set of neighboring agents, $k = |\mathcal{N}_i(t)|$ is cardinal number of this set, which is the fixed—given the considered topological nature of the interaction between agents, and $\eta_\mathrm{n}\xi_i(t)$ is a δ-correlated Gaussian noise taken in the $(-\pi, \pi)$ interval. Equation (4) is a sampled-data system, which is an archetypal model: (a) consistent with a host of empirical evidence gathered for biological swarms, and (b) yielding group-level ordering of the multiagent system—the order here refers to the degree of homogeneity, throughout the collective, of the state variables involved in the consensus reaching process—throughout entire flocks [37].

The time-dependent adjacency matrix represents the switching interaction network and is classically represented by the matrix $\mathbf{A}(t)$ defined by

$$\mathbf{a}_{ij}(t) \doteq \begin{cases} 1 & \text{in the presence of an edge from vertex } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that since temporary adaptive and directed graphs are considered here, the matrix $\mathbf{A}(t)$ is nonsymmetric and time-varying. The outdegree graph Laplacian of the IN fully embodies the topology of the communication channel at the group level:

$$\mathbf{L}(t) \doteq \mathbf{D}(t) - \mathbf{A}(t),$$

with $\mathbf{D}(t)$ the switching degree matrix defined as

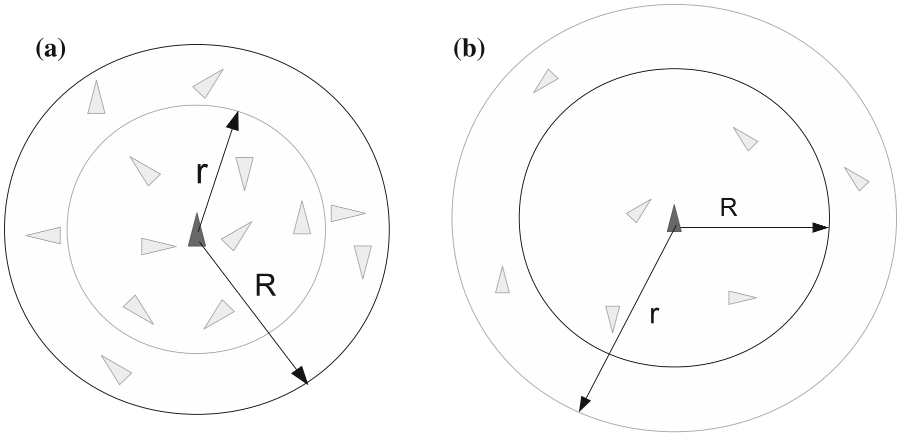


Fig. 2 Comparison of two common interaction distances in swarms: metric and topological interaction distances. **a**, **b** show two possible scenarios depending on the density of agents: R is the neighborhood radius in a metric sense, while r represents the distance between the central agent (dark filled arrow) and its farthest topological neighbor. This diagram assumes a topological interaction with 7 closest agents

$$\mathbf{D}(t) \doteq \text{diag}(d_1, \ldots, d_N), \quad \text{with} \quad d_i = \sum_{j \neq i} \mathbf{a}_{ij}, i = 1, \ldots, N.$$

We further introduce a reduced graph Laplacian

$$\tilde{\mathbf{L}}(t) \doteq \frac{1}{k}\mathbf{L}(t), \tag{5}$$

with k being the number of established topological links. It is worth noting that although the number of neighbors k remains constant the actual set of neighbors changes over time; see Sect. 2.2. The dynamical system (4) can conveniently be recast in matrix form:

$$\boldsymbol{\Theta}(t + T_{\mathrm{n}}) = \mathbf{P}_{\mathrm{n}}(t)\boldsymbol{\Theta}(t) + \eta_{\mathrm{n}}\boldsymbol{\Xi}(t), \tag{6}$$

with $\boldsymbol{\Theta}(t) \doteq [\theta_1(t), \ldots, \theta_N(t)]^T$, $\boldsymbol{\Xi}(t) \doteq [\xi_1, \ldots, \xi_N]^T$ and

$$\mathbf{P}_{\mathrm{n}}(t) \doteq (\mathbf{I} - T_{\mathrm{n}}\tilde{\mathbf{L}}(t)),$$

are time-varying Perron matrices [23]. It is worth highlighting their critical dependence on the bandwidth B_{n} through T_{n} [see Eq. (3)].

System (6) fully embodies the complex dynamic interplay between the network topology on the one hand and the information flow on the other hand. One of the aims of this study is to characterize the bandwidth B_{n} or equivalently unit interval T_{n} sufficient to guarantee the stability of consensus dynamic (6) in the presence of various levels of noise η_{n} in the communication channel.

3 Minimum bandwidth ensuring stability in the absence of noise

We now specifically consider the informational bottleneck corresponding to scenario (d) in Sect. 2.1. To this aim, we study the consequence of increasing T_{n}, which by Nyquist's theorem amounts to reducing B_{n} (see Eq. (3)) It is worth adding that the analysis below remains valid when replacing a IN nodal value of the bandwidth B_{n} by its edge counterpart as in scenario (c) in Sect. 2.1.

As mentioned in Sect. 2.2, our agents perform a swarming behavior corresponding to a heading consensus. The state of this multiagent system is represented by the time-dependent vector $\boldsymbol{\Theta}(t) \doteq [\theta_1(t),\ \theta_2(t),\ \cdots,\ \theta_N(t)]^T$, which is updated according to the time update rule (4) at the agent level, and (6) at the system level. The presence of noise in Eq. (4) (rightmost term in the RHS term) is a clear impediment to an exact analysis of the dynamics of this collective. As a starting point, we consider the noiseless case ($\eta_{\mathrm{n}} \equiv 0$), and we denote by $\boldsymbol{\Theta}_0$, the initial state at $t = 0$. With this simplifying assumption, the temporal evolution of the system can be obtained from

$$\boldsymbol{\Theta}(t + mT_{\mathrm{n}}) = [\mathbf{P}_{\mathrm{n}}((m-1)T_n)\mathbf{P}_{\mathrm{n}}((m-2)T_n)\ldots \mathbf{P}_{\mathrm{n}}(0)]\,\boldsymbol{\Theta}_0. \tag{7}$$

As already mentioned, the time dependence of $\mathbf{P}_{\mathrm{n}}$ is a signature of the dynamic interplay between network structure and information flow at the center of this study.

At this stage, it is worth discussing the well-studied static case corresponding to fixed network structure. With a constant $\mathbf{P}_{\mathrm{n}} = \mathbf{P}_0$, the system's stability can easily be derived from the graph spectral properties of $\mathbf{P}_0$ [23]. Unfortunately, these well-known static results are of no help in the general dynamic case. In the presence of temporal and adaptive INs, new results for the stability with decreasing B_{n} are required.

Theorem 1 *Consider the time-dependent system for heading consensus in swarms*

$$\mathbf{\Theta}(t + T_n) = (\mathbf{I} - T_n\tilde{\mathbf{L}}(t))\mathbf{\Theta}(t) = \mathbf{P}_n(t)\mathbf{\Theta}(t), \tag{8}$$

a necessary and sufficient condition for the stability (in the finite or asymptotic sense) of (8)*, is that its stability at every instant* $t_j = jT_n$ *is guaranteed.*

To this aim, we use the notion of joint spectral radius $\tilde{\rho}$ defined as [3]

$$\tilde{\rho} := \limsup_{j\to\infty} \left(\max_{t'_1,\cdots,t'_j \in\{t_1,t_2,\ldots,t_m\}} \|\mathbf{P}_{\mathrm{n}}(t'_1)\cdots\mathbf{P}_{\mathrm{n}}(t'_j)\| \right)^{1/j} \tag{9}$$

to assess the convergence of an infinite product of $\mathbf{P}_{\mathrm{n}}(t'_j)$ as per Eq. (7). Note that $\tilde{\rho}$ is actually independent of the choice of a matrix norm (i.e., this is true for any norm but particularly easy to see if the norm is sub-multiplicative). Moreover, we mention that Theorem 1 deals with stability, which is stronger than consensus reaching. Stability means convergence to an equilibrium, but consensus only requires the difference between any two individual states tends to zero.

Proof Let us consider $\rho(\cdot)$ to be the spectral radius of a given matrix. By taking $t'_1 = \cdots = t'_j = t$ in Eq. (9) and invoking Gelfand's spectral radius formula, we have

$$\tilde{\rho} \geq \lim_{j\to\infty} \|\mathbf{P}_{\mathrm{n}}(t)^j\|^{1/j} = \rho(\mathbf{P}_{\mathrm{n}}(t)), \tag{10}$$

for any $t \in \{t_1, t_2, \ldots, t_m\}$. Therefore, if $\rho(\mathbf{P}_{\mathrm{n}}(t)) > 1$, for any t, then $\tilde{\rho} > 1$. On the other hand, for any $\varepsilon > 0$, there exists a matrix norm $\|\cdot\|$ such that (e.g., [13, Lemma 5.6.10])

$$\begin{aligned}
\tilde{\rho} &\leq \lim_{j\to\infty} \left(\max_{t'_1,\cdots,t'_j \in\{t_1,t_2,\ldots,t_m\}} \|\mathbf{P}_{\mathrm{n}}(t'_1)\|^{1/j} \right)\cdots \\
&\quad \cdot \left(\max_{t'_1,\cdots,t'_j \in\{t_1,t_2,\ldots,t_m\}} \|\mathbf{P}_{\mathrm{n}}(t'_j)\|^{1/j} \right) \\
&= \max_{t\in\{t_1,t_2,\ldots,t_m\}} \|\mathbf{P}_{\mathrm{n}}(t)\| \\
&\leq \max_{t\in\{t_1,t_2,\ldots,t_m\}} \{\rho(\mathbf{P}_{\mathrm{n}}(t)) + \varepsilon\}.
\end{aligned} \tag{11}$$

Therefore, if $\rho(\mathbf{P}_{\mathrm{n}}(t)) < 1$ for all t, we can choose ε small enough so that $\tilde{\rho} < 1$. Recall that the system (8) is stable if and only if $\tilde{\rho} < 1$ [3,15]. □

Remark 1 Theorem 3.10 by Ren and Beard [26] (directly) investigates the graph theoretical conditions guaranteeing the consensus of time-dependent systems under some assumptions, while our result (indirectly) relates the stability condition for the time-dependent systems to that of the static systems at each switching point. This result has far-reaching consequences for the actual stability of the system, in terms of heading consensus for the swarm depending on the value of B_{n} as presented in the next Corollary.

Corollary 1 *For the NCS* (8) *to be stable, the following sufficient condition must be met in the form of an upper bound for the unit interval* T_n*:*

$$T_n < \frac{2}{\max_{1\leq i\leq N} |\lambda_i(\tilde{\mathbf{L}}(t))|} \quad \text{for all } t, \tag{12}$$

and this, at each instant in time $t_j = jT_n$*. The set* $\left\{\lambda_i(\tilde{\mathbf{L}}(t))\right\}$ *consists of all eigenvalues of the time-dependent reduced graph Laplacian* $\tilde{\mathbf{L}}(t)$ *of the network connectivity* [*see Eq.* (5)].

Proof From (12), it follows that for all i and t, $0 \leq \lambda_i(T_{\mathrm{n}}\tilde{\mathbf{L}}(t)) < 2$ and hence $-1 < \lambda_i(\mathbf{P}_{\mathrm{n}}(t)) \leq 1$. Note that we are unable to sharpen the upper bound to $\lambda_i(\mathbf{P}_{\mathrm{n}}(t)) < 1$ (hence, we cannot conclude the stability immediately) since 0 is always an eigenvalue of $T_{\mathrm{n}}\tilde{\mathbf{L}}(t)$ for any $T_{\mathrm{n}} \geq 0$. However, we know that as $T_{\mathrm{n}} \to 0$ the corresponding continuous system is stable. Therefore, when T_{n} becomes small enough (as specified by (12)), our system (8) is also stable and $\tilde{\rho} < 1$ follows. The above Theorem therefore allows us to conclude the proof of this Corollary. □

Using the upper-bound constraint (12) on $T_{\mathrm{n}} = 1/(2B_{\mathrm{n}})$, one can readily establish a sufficient condition on the node bandwidth B_{n} , which takes the form of the following lower-bond constraint

$$B_{\mathrm{n}} > B_{\mathrm{n}}^0 = \frac{1}{4} \max_{1\leq i\leq N} |\lambda_i(\tilde{\mathbf{L}}(t))| \quad \forall t. \tag{13}$$

In other words, the convergence to consensus of the multiagent system is assured by having the sufficient condition (13) be met. At this stage, it is worth highlighting two important facts: (1) the lower-bound value B_n^0 in Eq. (13) is obtained under the noiseless approximation, and (2) its value is not known *a priori*.

4 Combined effect of reduced data rate and noise on consensus reaching

As a next step, we study the more realistic case where the system's dynamics is affected by both reduced bandwidth and various noise levels in the communication channel. As far as we know, the combined effect of noise and limited bandwidth on consensus dynamic has never been investigated. Given the stiff nature of this problem, it is unlikely that analytical results equivalent to (13) for the minimum channel capacity C_n can be established. We therefore resort to a systematic numerical analysis based on simulations of a system of N agents whose dynamics is dictated by the discrete-time update rule (4). Specifically, we consider the combined influence of increasing T_n and η_n on the dynamics of this system, which is strictly equivalent to reducing the bandwidth [see Eq. (3)] in the presence of noise. The convergence in multiagent coordination is quantified by the polarization

$$\varphi(t) \doteq \left| \frac{1}{N} \sum_{p=1}^{N} \mathrm{e}^{\mathrm{i}\theta_p(t)} \right|,$$

a.k.a order parameter, and which is a good metric for the collective agreement—in this case of collective motion, agreement in the direction of travel—within the multiagent system. Essentially, a value $\varphi = 1$ corresponds to perfect consensus among agents, while $\varphi = 0$ denotes complete disagreement or disorder.

For large bandwidths, $B_\mathrm{n} \gg B_\mathrm{n}^0$, systems of vastly different sizes systematically achieve stability with high levels of global order, regardless of the noisiness of the node (see Fig. 3). When decreasing B_n below B_n^0, the swarm systematically undergoes a transition toward a globally disordered state—associated with a lack of consensus between agents—regardless of the population size N, and noisiness of nodes η_n (see Fig. 3).

As can clearly be observed in Fig. 3, the vanishing of consensus is a gradual process when reducing the information capacity (low B_n and/or high η_n). This continuous transition from a consensus reaching system to non-stabilizable one is better fathomed using the concept of phase transition borrowed from statistical physics. It is interesting to note that from the control theory standpoint, the system undergoes a transition from a stable collective state to an unstable one. However, effectively this transition process is continuous.

Indeed, we identify only continuous phases transitions as attested by the positive values of the Binder cumulant [4], $U \equiv 1 - \langle\varphi^4\rangle/3\langle\varphi^2\rangle^2$, when varying the bandwidth B_n as is clearly shown in Fig. 4a. Even for such continuous phase transitions, a critical value of the control parameter—bandwidth or noise—exists and can be determined numerically.

Given the expression (1) for the capacity in terms of bandwidth and $\mathcal{S}$ (i.e., noise), we expect the critical values B_n^c and η_n^c of bandwidth and noise, respectively, not to be independent but instead to be related through the existence of a unique critical capacity C_n^c. Here, we observe for the first time the appearance of a critical line characterized by $B_\mathrm{n}^c = B_\mathrm{n}^c(\eta_\mathrm{n})$, which is theoretically predicted. Indeed, the variance of the order parameter gives a sensible measure of the responsiveness of this multiagent system, which is related to the susceptibility—mathematically defined as $\chi \equiv \ell^2(\langle\varphi^2\rangle - \langle\varphi\rangle^2)$, whose dependency on the bandwidth is clearly peaked near criticality (see Fig. 5).

Still using the SH theorem and the dependence of the maximum capacity C_n on both B_n and η_n arising from Eq. (1), one can make some notable predictions about the identification of a critical line and its related properties. On this critical line $B_\mathrm{n}^c = f(\eta_\mathrm{n})$, where f is an unknown monotonic function, we have that B_n^c decreases with decreasing η_n. This predicted and observed trend along the critical line (see Fig. 5) is in tune with our expectations that more information exchanges are necessary in the face of higher levels of noise. Let us assume there exists a critical rate of data flow D^c such that if $C_\mathrm{n} < D^c$, then the self-organizing process is hampered and no group-level ordering emerges. At criticality, the MFMC theorem [24] takes the simple form $C_\mathrm{n} = D^c$, which imposes that $B_\mathrm{n}^c \downarrow$ when $\eta_\mathrm{n} \downarrow$ given Eq. (1). Without getting into the technical challenges associated with the unrealistic noiseless case of the SH capacity, we are able to numerically exhibit the existence of B_n^c thanks to the variations of the Binder cumulant with the bandwidth for different values of the noise (see Fig. 4b). As is classically known, the variations of the Binder cumulant with one control parameter—here B_n—for various val-

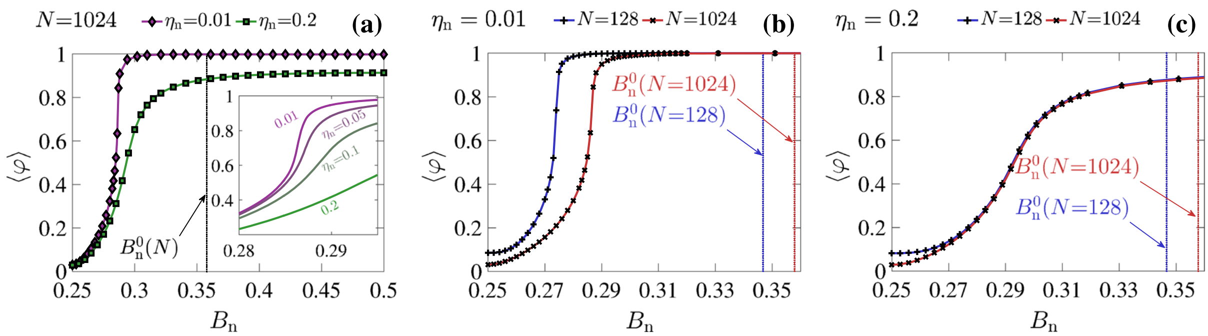


Fig. 3 Collapse in consensus—measured by $\langle\varphi\rangle$—with decreasing bandwidth B_n, various system's sizes and noise levels η_n. **a** $N = 1024$; **b** $\eta_n = 1\%$; **c** $\eta_n = 20\%$. Values for $B_n^0(N)$ are obtained from (13) within the noiseless limit and sampling over 10^4 INs. The following parameters were used: speed $v_0 = 0.3$, number of topological neighbors $k = 7$, and density $\rho = N/\ell^2 = 100$

Fig. 4 Binder cumulant at steady state: **a** extended interval of values for B_n; **b** near criticality $B_n^c = 0.286 \pm 0.001$. The following parameters were used: speed $v_0 = 0.3$, number of topological neighbors $k = 7$, and density $\rho = N/\ell^2 = 100$

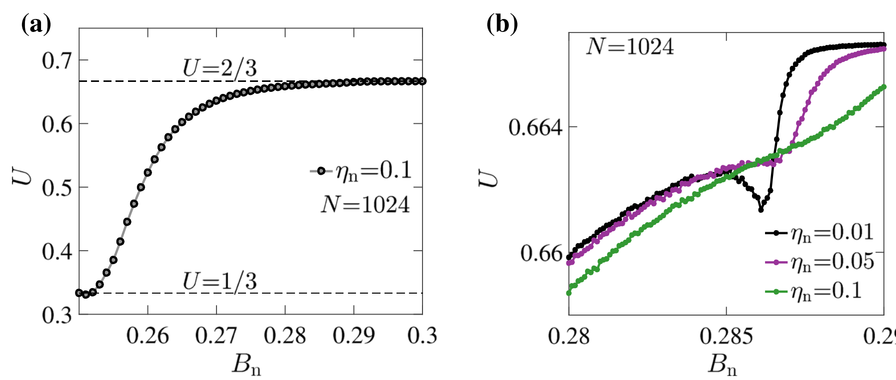
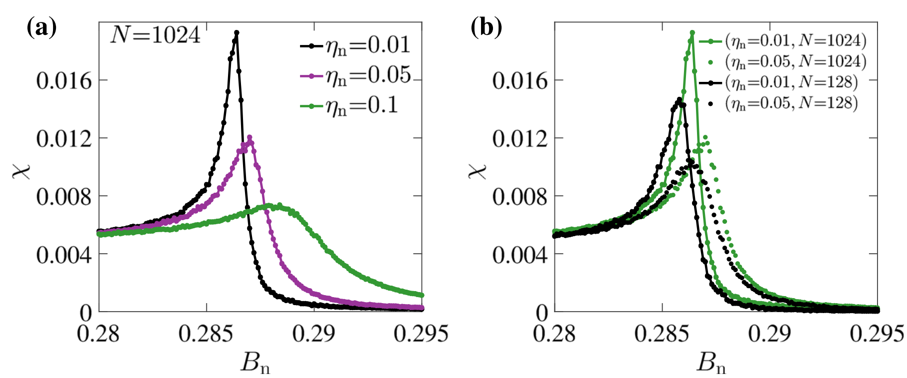


Fig. 5 Susceptibility at steady state for different swarm population sizes: **a** $N = 2^{10}$; **b** $N = 2^7$ and $N = 2^{10}$. The following parameters were used: speed $v_0 = 0.3$, number of topological neighbors $k = 7$, and density $\rho = N/\ell^2 = 100$



ues of another control parameter—here η_n—intersect at criticality [4].

Given that our analysis was based on numerical simulations, it is worth adding that all the above observations remain unchanged for larger values of N, other values of the density $\rho = N/\ell^2$, for a wide range of v_0, and for other values of $k > 7$.

5 Effect of time-delays on consensus dynamics

We now turn to the study of the effects of time-delays on the stability of topologically interacting mobile agents seeking consensus under time-varying topologies. Like previously, we assume that each agent is connected to k topological neighbors via information interaction, and the dynamical update rule for any agent i without time-delay takes the classical form

$$\dot{\theta}_i(t) = \frac{1}{k} \sum_{j \in \mathcal{N}_i(t)} \left\{ \theta_j(t) - \theta_i(t) \right\}, \tag{14}$$

which is the continuous-time equivalent of (4) in the absence of noise. Taking a system-level approach based on the IN, the system's dynamics takes a compact vector form

$$\dot{\mathbf{\Theta}}(t) = -\tilde{\mathbf{L}}(t)\mathbf{\Theta}(t), \tag{15}$$

with $\tilde{\mathbf{L}}(t)$ being the time-dependent reduced graph Laplacian introduced in Sect. 2.2 as $\mathbf{L}(t)/k$.

Assuming that communication delays exist between agents, (14) becomes

$$\dot{\theta}_i(t) = \frac{1}{k} \sum_{j \in \mathcal{N}_i(t)} \left\{ \theta_j(t-\tau) - \theta_i(t) \right\}, \tag{16}$$

where $\tau > 0$ is the constant delay. The matrix formulation for the dynamical system (16) reads as

$$\dot{\mathbf{\Theta}}(t) = -\mathbf{I}\mathbf{\Theta}(t) + \frac{1}{k}\mathbf{A}(t)\mathbf{\Theta}(t-\tau), \tag{17}$$

where $\mathbf{A}(t)$ is the adjacency matrix of the interaction network, and $\mathbf{I} \in \mathbb{R}^{N\times N}$ is the identity matrix. Noting that $\mathbf{\Theta}(t-\tau) = \mathbf{\Theta}(t) - \int_{t-\tau}^{t} \dot{\mathbf{\Theta}}(s)ds$, we can rewrite (17) as

$$\dot{\mathbf{\Theta}}(t) = \left(-\mathbf{I} + \frac{1}{k}\mathbf{A}(t)\right)\mathbf{\Theta}(t) - \frac{1}{k}\mathbf{A}(t)\int_{t-\tau}^{t} \dot{\mathbf{\Theta}}(s)ds, \tag{18}$$

where $-\mathbf{I} + \frac{1}{k}\mathbf{A}(t) = -\tilde{\mathbf{L}}(t)$ is the reduced Laplacian matrix defined above. Since $\mathbf{1}$ is the all-one eigenvector related to the zero-eigenvalue of $\mathbf{L}$, we can make a coordinate transformation $\mathbf{\Theta}(t) = \mathbf{W}(t)\mathbf{x}(t)$ such that

$$\mathbf{U}(t)\left(-\mathbf{I} + \frac{1}{k}\mathbf{A}(t)\right)\mathbf{W}(t) = \begin{bmatrix} \mathbf{B}(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{19}$$

where $\mathbf{B}(t) \in \mathbb{R}^{(N-1)\times(N-1)}$, $\mathbf{U}(t) \doteq (\mathbf{U}_1^T(t), \mathbf{U}_2^T(t))^T = \mathbf{W}^{-1}(t)$, with $\mathbf{U}_2(t) \in \mathbb{R}^{1\times N}$ being the last row of $\mathbf{U}(t)$. We first show that the system (18) is readily written using the new state $\mathbf{x}(t)$ by using the coordinate transformation $\mathbf{\Theta}(t) = \mathbf{W}(t)\mathbf{x}(t)$.

Lemma 1 *The system (18) can be rewritten as*

$$\dot{\mathbf{x}}_1(t) = -\mathbf{I}\mathbf{x}_1(t) + (\mathbf{B}(t)+\mathbf{I})\mathbf{x}_1(t-\tau), \tag{20}$$

$$\dot{\mathbf{x}}_2(t) = -\mathbf{x}_2(t) + \mathbf{x}_2(t-\tau), \tag{21}$$

where $\mathbf{x}(t) \doteq (\mathbf{x}_1(t)^T, \mathbf{x}_2(t))^T$ *with* $\mathbf{x}_1(t) \in \mathbb{R}^{N-1}$ *and* $\mathbf{x}_2(t) \in \mathbb{R}$.

Proof Substituting $\mathbf{\Theta}(t) = \mathbf{W}(t)\mathbf{x}(t)$ in (18) and multiplying both sides by $\mathbf{U}(t) = \mathbf{W}^{-1}(t)$ results in

$$\dot{\mathbf{x}}(t) = \mathbf{U}(t)\left(-\mathbf{I} + \frac{1}{k}\mathbf{A}(t)\right)\mathbf{W}(t)\mathbf{x}(t) - \frac{1}{k}\mathbf{U}(t)\mathbf{A}(t)\mathbf{W}(t)\int_{t-\tau}^{t} \dot{\mathbf{x}}(s)ds. \tag{22}$$

Using (19) and considering the point that $\frac{1}{k}\mathbf{U}(t)\mathbf{A}(t)\mathbf{W}(t) = \begin{bmatrix} \mathbf{B}(t)+\mathbf{I} & 0 \\ 0 & 1 \end{bmatrix}$, we arrive at

$$\begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{B}(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} - \begin{bmatrix} \mathbf{B}(t)+\mathbf{I} & 0 \\ 0 & 1 \end{bmatrix}\int_{t-\tau}^{t} \dot{\mathbf{x}}(s)ds. \tag{23}$$

Due to the diagonal structure of matrices in (23) and noting that $\int_{t-\tau}^{t} \dot{\mathbf{x}}(s)ds = \mathbf{x}(t) - \mathbf{x}(t-\tau)$, $\dot{\mathbf{x}}_1(t)$ and $\dot{\mathbf{x}}_2(t)$ are simplifies as

$$\begin{cases} \dot{\mathbf{x}}_1(t) &= \mathbf{B}(t)\mathbf{x}_1(t) - (\mathbf{B}(t)+\mathbf{I})\left(\mathbf{x}_1(t) - \mathbf{x}_1(t-\tau)\right) \\ \dot{\mathbf{x}}_2(t) &= -\mathbf{x}_2(t) + \mathbf{x}_2(t-\tau) \end{cases},$$

which lead to (20) and (21) and prove the statement of the Lemma. □

Clearly, by definition, we have $\mathbf{x}_2(t) = \mathbf{U}_2(t)\mathbf{\Theta}(t)$. If the interaction network is undirected, then $\mathbf{U}(t) = \mathbf{W}^T(t)$ and $\mathbf{B}(t)$ can be taken as a diagonal matrix. If the last column of $\mathbf{W}(t)$ is $a\mathbf{1}$, then $\mathbf{U}_2(t) = \mathbf{U}_2 = \frac{1}{aN}\mathbf{1}$ is time-invariant.

The following main result characterizes the stability of consensus affected by time-delay.

Theorem 2 *Assume that the interaction network is always strongly connected. Considering the system (17) with time-delay* τ*. If there exists* $\bar{\tau} \geq \tau$ *such that*

(i) $1 + \lambda(-\mathbf{B}(t) - \mathbf{I})\frac{1-e^{s\bar{\tau}}}{s} \neq 0$ *holds for all* $t \geq 0$, $s \in \mathbb{C}^+$, *and eigenvalues of* $-\mathbf{B}(t) - \mathbf{I}$*; and*

(ii) there exist two positive-definite matrices $(\mathbf{P}, \mathbf{Q})$*, for which the time-dependent linear matrix inequality (LMI) below holds:*

$$\begin{bmatrix} \mathbf{B}^T(t)\mathbf{P} + \mathbf{P}\mathbf{B}(t) + \bar{\tau}\mathbf{Q} & \mathbf{B}^T(t)\mathbf{P}(\mathbf{B}(t)+\mathbf{I}) \\ \bullet & -\bar{\tau}\mathbf{Q} \end{bmatrix} < 0, \tag{24}$$

where $\bullet$ *denotes entries that come from symmetry,*

then, the system (17) reaches a consensus.

Proof See Appendix 7.1. □

Note that the above LMI is feasible for each given t, meaning that for small enough $\bar{\tau}$, there always exist positive-definite matrices $\mathbf{P}$ and $\mathbf{Q}$ so that the dynamic LMI (24) above holds. In fact, given t, for any positive-definite matrix $\mathbf{Q}$ and a small enough $\bar{\tau} > 0$, the matrices $\mathbf{B}^T(t)\mathbf{P} + \mathbf{P}\mathbf{B}(t) + \bar{\tau}\mathbf{Q}$ and $-\bar{\tau}\mathbf{Q}$ are negative definite. One can then choose a suitable positive-definite $\mathbf{P}$ (e.g., a diagonal matrix with small but positive diagonal elements) to satisfy Eq. (24). For changing t, the above $\mathbf{Q}$ and the "minimum" of the above $\mathbf{P}$ matrices suffice. We can use the bound of eigenvalues of $\mathbf{B}(t)$ as indicated in Appendix 7.1. Therefore, no large computations are required even though the LMI is time-dependent.

Additionally, Theorem 2 has some important practical implications for the coordination of swarm robotics systems. Indeed, as recently reported in Refs. [8, 41], swarm robotics systems operating based on distributed communications through a mesh network will inevitably be affected by time-delays. Theorem 2 implies that by properly designing the mesh network (e.g., see Ref. [29]), one can expect to achieve consensus and swarm coordination if time-delays are appropriately brought below a certain level.

Lastly, it is interesting to note the similarities between the present work on consensus reaching systems of topologically interacting agents and synchronization problems of various neural networks with sampled-data controller in the presence of time-delays [25,39,40]

6 Conclusion

Recent empirical studies have highlighted the critical importance of robust and accurate transfer of information among individuals engaged in swarming behaviors. These results stress the possible adverse effects of agents afforded with limited sensory capabilities or evolving in noisy environments, which combined may hinder self-organization.

The study of these limiting effects on the consensus dynamics of a networked multiagent system has been carried out using two complementary approaches. First, we neglected the effects of noise in Eq. (6) and modeled the swarm as a NCS governed by Eq. (8). The study of the stability and asymptotic stability of the system led to Eq. (13). A sufficient condition for the node bandwidth B_{n} is established in the form of an upper bound, which brings to light the key interplay between: (a) the dynamic network topology—by means of the largest eigenvalue of the reduced graph Laplacian of the interaction network $\tilde{\mathbf{L}}(t)$, and (b) the minimum social information transfer, which ensures global ordering of the swarm—by means of the node bandwidth. It is worth adding that although our study focuses on a particular type of consensus problem—namely consensus in the direction of motion of agents traveling at constant speed while topologically interacting—the obtained results can be extended to other collective behaviors of the consensus type. Specifically, the methodology and approach developed here, with switching topologies in the presence of communication constraints, can be extended to any linear distributed decision-making process of the consensus type, including formation control, rendez-vous in space, aggregation, etc. However, second-order linear or nonlinear consensus algorithms—e.g., guaranteed-cost consensus for multiagent networks with Lipschitz nonlinear dynamics and switching topologies [36]—would require completely different studies to account for the issues of communication constraints in the presence of switching topologies.

This analytical result, obtained in the noiseless case, is supplemented by simulations of a swarm of self-propelled particles with increasingly small bandwidth and for various levels of noise. These simulations are instrumental in investigating the swarm dynamics near criticality, i.e., in the vicinity of the continuous phase transition separating a globally ordered swarm, with a stable dynamics, from a disordered one corresponding to an unstable dynamics. To the best of our knowledge, all prior studies of SPP dynamics were obtained with bandwidths significantly above the critical level identified here. In these past simulations, the disappearance of collective motion always found its source in the high level of noise and/or low density of agents. In all cases considered, the breakdown in the self-organization is observed and consistent with the noiseless analytical study. Our analysis also uncovers the presence of a critical line with a particular monotonic relationship between critical bandwidth and noise that is consistent with classical results in information theory.

Furthermore, we investigated the influence of communication time-delays on the consensus reaching process for multiagent systems whose interactions are governed by a topological distance leading to time-varying network topologies. Specifically, we found the maximum allowed time-delay below which consensus is guaranteed.

Lastly, the analytical results obtained in this paper are currently being tested with a swarm robotics system made of 15 units of a small-size differential-drive land robot, equipped with a host of sensors, and capable of moving up to a speed of 20 cm/s. These robots are interconnected by means of a swarm-enabling unit [8], which allows us to implement a topological interaction rule and also indirectly tune time-delays between swarming units.

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Compliance with ethical standards

Conflicts of interest The authors declare that there is no conflict of interest regarding the publication of this article.

7 Appendices

7.1 Proof of Theorem 2

The following Lemma is instrumental to establish the proof of Theorem 2.

Lemma 2 *Consider the following linear system*

$$\begin{cases} \dot{x}(t) = \mathbf{A}^0 x(t) + \mathbf{A}^1(t)x(t-\tau), \\ \mathbf{x}(s) = \boldsymbol{\phi}(s), \forall s \in [-\tau, 0] \end{cases},$$

where $\mathbf{x} \in \mathbb{R}^N$ is the state variable, $\mathbf{A}^0$ is a constant matrix and $\phi(s)$ corresponds to the set of initial conditions considered over the interval $[-\tau, 0]$. The system is asymptotically stable for all values τ in the interval $[0, \bar{\tau}]$ if

(i) $1+\lambda(\mathbf{A}^1(t))\frac{1-e^{s\bar{\tau}}}{s} \neq 0$ *holds for all $t \geq 0$, $s \in \mathbb{C}^+$, and $\lambda(\mathbf{A}^1(t))$ are the eigenvalues of $\mathbf{A}^1(t)$; and*

(ii) there exist two positive-definite matrices $(\mathbf{P}, \mathbf{Q})$ such that the LMI below holds:

$$\begin{bmatrix} (\mathbf{A}^0 + \mathbf{A}^1(t))^T\mathbf{P} + \mathbf{P}(\mathbf{A}^0 + \mathbf{A}^1(t)) + \bar{\tau}\mathbf{Q} & (\mathbf{A}^0 + \mathbf{A}^1(t))^T\mathbf{P}\mathbf{A}^1 \\ \bullet & -\bar{\tau}\mathbf{Q} \end{bmatrix} < 0.$$

Proof It can be proved as in [22, p. 222] by considering the following Lyapunov candidate

$$\mathbf{V}(t) = \left(\mathbf{x}^T(t) + \int_{t-\tau}^{t} \mathbf{x}^T(s)ds\mathbf{A}^1(t)^T\right)\mathbf{P} \cdot \left(\mathbf{x}(t) + \mathbf{A}^1(t)\int_{t-\tau}^{t} \mathbf{x}(s)ds\right) + \int_{-\tau}^{0}\int_{t+r}^{t} \mathbf{x}^T(s)\mathbf{Q}\mathbf{x}(s)dsdr.$$

□

We now prove the statement of Theorem 2. First, note that all eigenvalues of the matrix $\mathbf{B}(t)$ are located in the open-right plane given that the interaction network is strongly connected. Therefore $\mathbf{B}^T(t)\mathbf{P}+\mathbf{P}\mathbf{B}(t)$ can be negative definite, and the linear-matrix inequality (24) is feasible for small enough $\bar{\tau}$. Next, using Lemma 1, stability of the systems (17) or (18) is equivalent to the stability of (20) and (21). By exploiting Lemma 2 with $\mathbf{A}^0 = -\mathbf{I}$ and $\mathbf{A}^1(t) = \mathbf{B}(t) + \mathbf{I}$, it is observed that the solution of system (20) tends to zero, i.e., $\lim_{t\to\infty} \mathbf{x}_1(t) = 0$.

Finally, the Laplace transform of (21) yields

$$\mathbf{X}_2(s) = \frac{\mathbf{x}_2(0) + \int_{-\tau}^{0} \mathbf{x}_2(u)e^{-(u+\tau)s}du}{s+1-e^{-\tau s}},$$

where s is a complex variable. The stability of (21) is defined by the denominator of the Laplace transform ($s+1-e^{-\tau s}=0$). Letting $s = \sigma + j\omega$ with $\sigma, \omega \in \mathbb{R}$ and $j^2 = -1$, we have

$$\sigma + 1 - e^{-\sigma\tau}\cos(\omega\tau) = 0, \tag{25}$$

$$\omega + e^{-\sigma\tau}\sin(\omega\tau) = 0. \tag{26}$$

The system (21) is stable for any time-delay τ, i.e., there exists a $\mathbf{x}_2^*$ such that $\lim_{t\to\infty} \mathbf{x}_2(t) = \mathbf{x}_2^*$ if solution of (25) leads to $\sigma < 0$. We consider the following cases:

(i) $\sigma = 0 \Rightarrow \cos(\omega\tau) = e$ which is impossible since $-1 \leq \cos(\omega\tau) \leq 1$.

(ii) $\sigma > 0 \Rightarrow e^{-\sigma\tau}\cos(\omega\tau) = \sigma + 1 > 1$ which is again impossible since $e^{-\sigma\tau} < 1$ and $-1 \leq \cos(\omega\tau) \leq 1$, then their product cannot be larger than 1.

Therefore, the value of σ obtained from (25) is strictly negative ($\sigma < 0$) which guarantees the stability of system (21). This concludes our proof.

7.2 Remarks about Theorem and Corollary 1 and Theorem 2

It is worth noting that Theorem 1 and Corollary 1 are based on the Gelfand spectral radius formula and on the construction of the joint spectral radius. The special structure of the matrix $\tilde{\mathbf{L}}(t) = L(t)/k$, originating from the topologically interacting agents, plays a key role here. These considerations are very different from the result in Ref. [26].

The key challenge in proving Theorem 2 mainly lies in the construction of the LMI and the appropriate application of the Laplace transform. Another key simplification is obtained by means of the coordination transformation in Eq. (19), which simplifies our equation and facilitates the analytical treatment for the time-delay system.

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