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Study of the indirect interaction in a non-Fermi liquid within the AdS/CFT correspondence framework

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The indirect spin–spin interaction between impurities in a non-Fermi quantum liquid system is theoretically investigated in this paper. The poles of the Green's functions are shown to be responsible for the observed excitation spectra. Specifically, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence is used to gain access to the analytical expressions of the Green's functions for our particular problem.

Keywords: Indirect interaction; anti-de Sitter/conformal field theory correspondence; non-Fermi quantum liquid.

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1. Introduction

Recently, it has been shown that many problems rooted in solid state physics — particularly those related to quantum critical phenomena — could be solved by borrowing certain ideas from particle physics. More specifically, the relevance of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence formalism for this family of phenomena has been underscored.[1,2]

In particular, the AdS/CFT correspondence between the theory of superstrings and the CFT provides a significant progress as it allows to gain access to the analytical expressions of the Green's functions for various systems.[7–9] In turn, accounting for the fact that solid state physics provides numerous experimental methods and tools, some of the essential general consequences associated with the AdS/CFT

theory can thereby be indirectly tested and verified. Indeed, advances in condensed matter physics, where inter-particle interactions play a decisive role, have been achieved in parallel with the progress made in the physics of strong interaction. At some point in the development of the theory for the strong interaction — a.k.a. quantum chromodynamics, theorists have faced significant computational difficulties. Numerous attempts to overcome the latter have brought two leading players in the field, namely the CFT and the theory of strings/superstrings. The string theory has been successful in circumventing certain essential problems by formally and conveniently increasing the space dimension. The CFT is conventionally dealing with the Green's functions. However, under specific conditions, these Green's functions can take a simplified expression. The primary advantage of the AdS/CFT correspondence theory is that it reveals a profound connection between the two above-mentioned branches, both ideologically and technically.[10]

While recent some research activity in this area has been primarily focused on the conduction phenomena,[11] there are still many other characteristics that can be tested and verified experimentally. Among others, one of these characteristics is the value of the indirect interaction of the impurity spins, which reflects the contribution to the energy of the system depending on the state of the impurities' spins. A compelling example of such an interaction is given by the well studied Ruderman–Kittel–Kasuya–Yosida (RKKY) interaction,[12,13] when the impurity spins interact via the conduction electrons. The physical origin for this interaction is quite transparent and is associated with a change in the state of the impurity spin when the electron scattering event occurs after the preceding scattering event with some other impurity. The technical procedure of introducing an interaction of this kind is however controversial to some extent. Indeed, it consists in the projection of the space of states of two subsystems (e.g. electrons and spins in the case of the RKKY interaction) onto the state space of one subsystem (the spin system in the presently discussed example). An alternative approach is given by the Fröhlich formalism, which is discussed and presented in this paper. Within the latter framework, we initially perform some canonical transformations, and only then proceed with the projection. The results of both of the above-mentioned approaches are qualitatively consistent, and in addition, both have been successfully applied to the study of real physical systems — e.g. in the Bardeen–Cooper–Schrieffer (BCS) theory.[14,15]

Thus, the description of the indirect interactions in quantum critical phenomena by borrowing the relations taken from AdS/CFT correspondence represents a promising goal at the heart of the present research reported in this paper.

2. Governing Equations

Let us start with the standard Hamiltonian, commonly applied to the exchange models:[16]

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}\,, \tag{1}$$

where

$$\mathcal{H}_0 = \sum_{k,\sigma} \epsilon_k a^\dagger_{k\sigma} a_{k\sigma}, \quad \mathcal{H}_{\rm int} = \sum_{p,p'} I(\mathbf{q}) \sum_{\sigma\sigma'} S_q \sigma_{\sigma\sigma'} a^\dagger_{p\sigma} a_{p'\sigma'}\,, \tag{2}$$

$S_q = \sum_R \exp(i\mathbf{q}\mathbf{R}S_R)$, $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, σ are the Pauli matrices, $a^\dagger_{k\sigma}$ and $a_{k\sigma}$ are the creation and annihilation operators for the excitations with the spin σ and the wave vector $\mathbf{k}$, related to the excitation energy through a dispersion law $\epsilon(\mathbf{k})$; $I(q)$ is the Fourier transform of the interaction potential between a d-impurity and electrons in the conduction band; S_R stands for the spin vector of an impurity situated at the point R. Note that this particular choice of the interaction Hamiltonian for excitations can be justified by firstly its simplicity (the proposed approach can be easily generalized to more complex cases) and, secondly, its invariance properties.

The Fröhlich method for the treatment of indirect interactions[14] is based on the assumption, that operators' matrix elements are consistent with the inequality $|\mathcal{H}_{\rm int}| \ll |\mathcal{H}_0|$ (in the sense of the eigenvalues within a basis of the operator $\mathcal{H}_0$). To proceed with the first stage of the Fröhlich method, we move to a new representation through the unitary transform $U = \exp(-\mathcal{L})$, where $\mathcal{L}$ is the anti-Hermitian operator satisfying the condition

$$\mathcal{H}_{\rm int} + [\mathcal{H}_0, \mathcal{L}] = 0\,. \tag{3}$$

Using this new representation, the Hamiltonian takes the form:

$$\mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{H}_0 + \frac{1}{2}[\mathcal{H}_{\rm int}, \mathcal{L}] + O(\mathcal{H}^3_{\rm int})\,. \tag{4}$$

The formal solution to the operator Eq. (3) reads

$$\mathcal{L} = \frac{1}{i\hbar} \lim_{\varepsilon\to 0} \int_{-\infty}^{0} e^{\varepsilon t} \mathcal{H}_{\rm int}(t) dt, \quad \mathcal{H}_{\rm int}(t) = \exp\left(-\frac{\mathcal{H}_0 t}{i\hbar}\right) \mathcal{H}_{\rm int} \exp\left(\frac{\mathcal{H}_0 t}{i\hbar}\right).$$

After calculation of the corresponding commutators, the operator $\mathcal{L}$ takes the form:

$$\mathcal{L} = \sum_{k_1,k_2} I(\mathbf{q}) \sum_{\sigma} \left\{ \frac{(S^x_q - iS^y_q) a^\dagger_{k_1,\sigma} a_{k_2,-\sigma}}{\epsilon_{k_2} - \epsilon_{k_1}} + \frac{(S^x_q + iS^y_q) a^\dagger_{k_1,-\sigma} a_{k_2,\sigma}}{\epsilon_{k_2} - \epsilon_{k_1}} \right\}. \tag{5}$$

Substituting Eq. (5) into Eq. (4) and taking the thermodynamic average gives us the spin–spin indirect interaction Hamiltonian in the form

$$\mathcal{H}_{SS} = \sum_{k_1,k_2,R_1,R_2} 4M(S^1_x S^2_x - S^1_y S^2_y) \cos(\Delta\mathbf{k}\Delta\mathbf{R})\,, \tag{6}$$

where we have introduced the following notations: $\mathbf{k}_1 - \mathbf{k}_2 = \Delta\mathbf{k}$, $\mathbf{R}_1 - \mathbf{R}_2 = \Delta\mathbf{R}$,

$$M = (1/2t) I_{\Delta k} I_{-\Delta k} \beta \exp\{-\beta\epsilon_{k_2}(e^{-t} - 1)\}, \quad t = \beta(\epsilon_{k_1} - \epsilon_{k_2})\,.$$

Let us choose the z-axis to be co-directed with the radius-vector linking the interacting particles. Keep in mind that in this geometry $\mathbf{R} = \{0, 0, R\}$, $\mathbf{k}_1 = \{0, k_1, \cos\theta_1\}$, and $\mathbf{k}_2 = \{0, k_2, \cos\theta_2\}$. Let us perform the explicit integration over the azimuthal angles in Eq. (6) and move from a discrete representation to the continuous one.

Within the Frölich framework, after the canonical transformation and taking the average, the quantity standing before spins in the Hamiltonian bears a sense of the indirect interaction constant. Thus, as a result, the indirect interaction constant becomes

$$J(R) = (2\pi)^2 \int_{-\Delta}^{\Delta} dk_1 \int_{-\Delta}^{\Delta} dk_2 M(k_1, k_2) k_1^2 k_2^2 \times \int_{-\pi/2}^{\pi/2} d\theta_1 \int_{-\pi/2}^{\pi/2} d\theta_2 \sin\theta_1 \sin\theta_2 \{(k_1 \cos\theta_1 - k_2 \cos\theta_2) R\}, \quad (7)$$

where Δ is a small area close to the Fermi level. Implicit dependence of the wave vector on the frequency of interaction, as well as an opportunity to compute integrals over the spherical angles of their own, allow us to rewrite the expression (7) in terms of frequency as follows:

$$J(R) = (2\pi)^2 \frac{8}{R^2} \int_{-\Delta}^{\Delta} d\omega_1 \int_{-\Delta}^{\Delta} d\omega_2 M(\omega_1, \omega_2) f'(\omega_1) f'(\omega_2) \times f(\omega_1) f(\omega_2) \sin[R f(\omega_1)] \sin[R f(\omega_2)], \quad (8)$$

where we have used the following notations

$$\epsilon(k_i) = \omega_i, \quad dk_i = f'(\omega_i) d\omega_i \quad (i = 1, 2).$$

Note that this dependence implicitly contains the oscillating part, so that evaluating integrals by the saddle-point method one can show the presence of decay, i.e. a strong decrease of the value of the indirect interaction with increasing distance between the impurities. Such a behavior always takes place for the mechanisms of indirect exchange via conduction electrons, and is due to the local character of the interaction between electrons and impurity atoms.

3. Excitation Spectrum from the AdS/CFT Correspondence

The function $k = f(\omega)$ represents the dispersion relation connecting the wave vector and the energy of the elementary excitations of the given substance in the given critical region. The most simple way to calculate it is by having a close look at the poles of the corresponding Green's function. For systems near the quantum critical point, namely, for the "strange metals", the Green's function can be found using the holographic approach. Such materials, and some high-temperature superconductors in the quantum critical region are dual models of the superstring theory with gravity.[16,17] We use the term "duality" in its commonly accepted sense, that is, in terms of mean values the theory of gravity with matter in space-time in a certain dimension is equivalent to the field theory on the boundary of space-time. This allows us to calculate the wave functions of the perturbation fields in the volume and immediately obtain expressions for the mean values of the dual variables

(understood in the usual sense of the CFT) on the boundary, and the Green's functions.[18,19] Thus, to calculate the Green's functions for the "strange metal" it was proposed in Ref. 20 to consider the space-time with the metric of the form:

$$ds^2 = \frac{f(r)}{r^{2z}}dt^2 - \frac{dt^2}{f(r)r^2} - \frac{d\mathbf{x}^2}{r^2}\,,$$

where $d\mathbf{x}^2$ stands for the rest part of metrics, r is the variable over which the transition to the border is performed. The function $f(r)$ always has a simple zero while r is approaching some value r_h, which is known as the horizon:

$$f(r)|_{r\to r_h} = c(1 - r/r_h)\,,$$

where c is some positive constant. Temperature in the model is defined through $T = cr_h^{-z}/4\pi$, and z stands for the critical exponent.[20–22] Note, that in present study we are not aiming to establish a connection with the superstring theory, but we rather use the results for the given metrics, obtained in the above-mentioned papers. Following the ideas developed in Refs. 8 and 20, we readily obtain that the poles of the Green's function are determined by the expression

$$k = C\sqrt{\omega^2 + \mathrm{const}\, G(\omega)}\,, \quad G(\omega) = \begin{cases} \omega^{1+2/z}, & z \neq 2 \\ \omega \log(a\omega), & z = 2 \end{cases}\,.$$

As shown in Ref. 20, quasiparticle excitations can exist only in the region $z < 2$, and in a 3D space they have the following spectrum

$$k = A\omega\sqrt{1 + r\omega^{-1+2/z}}\,. \tag{9}$$

Indeed, for $z < 2$, close to the Fermi level, the real part of the dispersion law behaves like k, while the imaginary part behaves like k^2. This fact allows us to claim the existence of elementary excitations.

4. Numerical Analysis and Conclusions

The dependence of the indirect interaction constant calculated from (8) on the distance for different values of the critical exponent is shown in Fig. 1. As it can be seen from the results, the critical exponent z strongly affects the dependence of the exchange interaction constant on the distance, causing the appearance of secondary local maxima. This, in turn, gives us the possibility to experimentally verify the AdS/CFT relations using, say, spin spectroscopy methods. For instance, this could be achieved by considering the spectrum of spin waves in the impurity system. As it is well known, due to the indirect interaction, an effective spin–spin interaction arises, resulting in the appearance of the spin waves.[23] The nature of these spin waves is similar to the spin waves in magnetic materials, given the fact that the interaction between the spins is indirect. Thus, the methods for their study are similar. For example, the test can be done by using the parametric excitation, when the frequency of the spin wave is recorded via the resonant absorption of the applied alternating field energy.

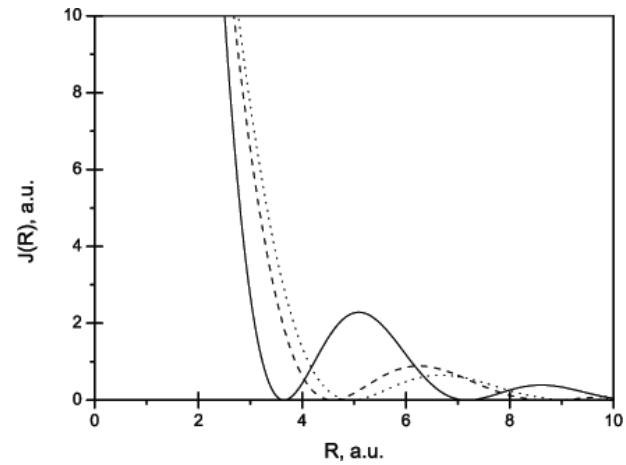

Fig. 1. The dependence of the constant of the indirect interaction (in arbitrary units a.u.) on the distance for different values of the critical exponent z: solid line $z = 0.8$; dashed line $z = 1$; dotted line $z = 1.5$.

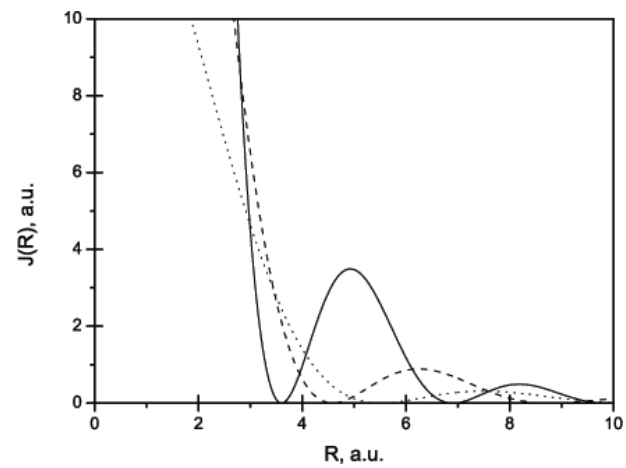

Fig. 2. The dependence of the constant of the indirect interaction (in arbitrary units a.u.) on the distance for different values of the parameter r ($z = 1$): solid line $r = 0.4$; dashed line $r = 1$; dotted line $r = 1.6$.

The dependence of the constant of the indirect interaction on the distance for different values of the parameter r is presented in Fig. 2. The basic conclusion following from our numerical analysis is the fact that the indirect interaction constant rather strongly depends on both independent parameters r and z. Again, this provides us with a unique possibility for experimental verifications of the AdS/CFT correspondence using the well developed methods of the solid state theory. First of all, the above correspondence allows us to correctly identify the area in which quasiparticles exist and to link their existence to the critical exponent.

In conclusion, the experimental study of the indirect interaction in non-Fermi quantum liquids — with a particular emphasis on measuring the indirect interaction constant — provides us with a possibility to test and verify certain underlying principles of the AdS/CFT theory. Note that our study is only valid when considering the area of "well defined" quasiparticles (i.e. the imaginary part of the Green's functions decreases faster than the real one and $z < 2$). The calculations of indirect interaction characteristics away from the Fermi level represent a separate and challenging problem altogether, which is beyond the scope of the present paper.

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