Robust Stabilization of a Class of Networked Nonlinear Systems via Parsimonious Communication and Actuation*

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***Abstract*—This paper proposes to design a robust controller for a class of nonlinear networked control systems using aperiodic feedback information. The parameter variation and system nonlinearity are considered as sources of uncertainty. To tackle uncertainty in system dynamics, a linear robust control law is derived using optimal control theory. Two different architectures of closed-loop systems are considered. In the first one, system and controller are not collocated; instead they are interconnected by means of a shared communication network. In the second architecture, however, sensors and controller are connected through a shared communication channel. In both architectures, the feedback loop is closed through the network. To save network bandwidth, the state and input information are transmitted aperiodically through the feedback loop. To this aim, the paper adopts an event-triggered control approach to reduce the transmission overhead. We show that the designed event-triggered controller achieves a trade-off between control performance and saving network bandwidth in the presence of uncertainty. The developed control algorithm is implemented and validated numerically on a classical nonlinear system.**

***Index Terms*—Robust-optimal control, Event-triggered control, Bandwidth limitations, Nonlinear systems, Input-to-state stability.**

I. Introduction

In most Cyber-Physical Systems (CPSs) or Networked Control Systems (NCSs), each physical system shares its own local information with other subsystems through a communication network. Given the shared nature of the communicating channel, controlling such systems with continuous or periodic control laws implies large bandwidth requirements [1], [2]. Recently, event-triggered control approaches have been introduced to reduce the information requirements while achieving a stable control strategy, see [3]–[7]. Specifically, in the event-based control framework, the violation of a prespecified event condition leads to specific sensing and actuation instants at the sensor/actuator ends.

This event-triggering law primarily depends on the system's present state or outputs. In the event-triggered control framework, the key issue with the continuous-time form is the stringent requirement in continuously monitoring the event condition occurrence. For instance, in [7], the monitoring of the event-triggering condition is conducted periodically. To overcome the need for such a continuous/periodic monitoring, a self-triggered control approach has been proposed in [8]. In this self-triggered control approach, the subsequent time instant for event occurrence is determined using the system's state or output information at the previous sampling instant. For both classical event-triggering and self-triggering controls, a reduction in the overall network use can be achieved by increasing the time interval between triggering events. In the specific context of CPSs and NCSs, the primary role played by aperiodic sensing and actuating—for continuous and periodic event-triggered control—has been reported in [1], [2].

The key deficiency with classical event-triggered control is the need to have access to an accurate model of the system in order to devise the event-triggering rule. In practice, system modeling inevitably simplifies the actual system and thereby introduces a certain level of inaccuracy, which has nontrivial practical implications. It is worth highlighting that there is a vast breadth of problems related to event-triggered control in the presence of uncertainty. Such uncertainty have several possible origins: nonlinearities, variations in the system's parameters, components unaccounted for in the dynamical model, and pervasive perturbations. These issues call for the development of a specific controller. Recently, a robust event-triggered control algorithm has been developed based on aperiodic feedback in order to deal with the presence of uncertainty, albeit limited to linear system [9], [10]. Specifically, Tripathy et al. have adopted an optimal control strategy to design such

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a robust control law [9], [10]. Originally, this control law has been developed in [11], [12] based on concepts belonging to optimal control theory. The nominal dynamics has been used to design the control law. To realize the robust control law in [9] and [10], a prior assumption is made such that the model of the system is considered to be linear in nature. But in practice, most systems are nonlinear. Therefore, considering nonlinear systems is a far more realistic and pertinent control problem. Moreover, extending robust control results mentioned in [9], [10] to a class of nonlinear systems in the presence of bandwidth constraints in the communication channel is not straightforward.

In this paper, a robust event-triggered control algorithm is proposed to stabilize a class of networked nonlinear systems with aperiodic feedback information. In the networked system we consider in this paper, the connection between sensor and controller is established via a communication link, see Figs. 1 and 2. To design controllers, we rewrite the system dynamics as a linear model plus uncertainty. With this formulation, the system nonlinearities and parametric variations are considered as a source of uncertainty. An event-based linear robust control algorithm is proposed to stabilize this class of nonlinear systems with aperiodic feedback information. To regulate the behavior of this system when faced with multiple sources of uncertainty, two different event-based control algorithms are introduced. To ensure the closed-loop stability of such systems, a robust control law is computed using the nominal dynamics and the prior knowledge of the uncertainty bound. In addition, the derived controller gain matrix is used to analyze the closed-loop performance. This paper considers the input-to-state stability (ISS) theory [14] for analysis. The key contributions of this work are listed below:

- The nonlinear dynamical systems is divided into two parts—the linear and nonlinear one. The nonlinear part and parameter variations of the system model are treated as a source of matched uncertainty. In the optimal control framework, the design of a robust controller is based on a linear control law derived from the solution of a modified Linear Quadratic Regulator (LQR) problem ensuring the closed-loop stability of the original nonlinear system. A novel event-triggering rule is developed to reduce the amount of information exchange required to stabilize this class of systems.
- We also propose a robust event-triggered controller for uncertain systems with optimal event-triggering. To solve the robust controller and optimal event-triggering law, a joint optimization problem is formulated by minimizing a cost-function. The cost-function embodies both control and communication costs for optimal usage of resources. It is shown that the design of the robust optimal event-triggered controller in the optimal control framework is split in two sub-problems—the design of robust controller using the modified (LQR) framework and the optimal event-triggering sequence using a dynamic programming method.
- A classical nonlinear problem is investigated numerically to assess the effectiveness of the approach.

Notations and Definitions: The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$, $\mathbb{R}^n$ refers to the vector space of real vectors of dimension n, and by extension, $\mathbb{R}^{n\times m}$ is the vector space of real-valued n-by-m matrices. The notation $\mathbb{R}_{\geq 0}$ refers to the set of non-negative real numbers. The symbols $A \leq 0$, A^T and A^{-1} are classically used to specify the negative semi-definite character of a matrix A, its transpose, and its inverse respectively. The symbol I denotes the identity matrix of appropriate dimension. The maximum (resp. minimum) eigenvalue of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is $\lambda_{\max}(P)$ (resp. $\lambda_{\min}(P)$). A continuous function f: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{K}_\infty$ if it is strictly increasing and $f(0) = 0$ and $f(s) \to \infty$ as $s \to \infty$. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{K}$, if it is continuous, strictly increasing and $f(0) = 0$. A continuous function $\beta(r,s) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$ function, if it is a class-$\mathcal{K}$ function with respect to r for a fixed s, and it is strictly decreasing with respect to s when r is fixed [16]. We remark that the definitions used throughout this paper are identical to those found in the literature [14], [16].

Definition 1 (Input-to-State Stability):
A continuous-time system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

is input-to-state stable (ISS) if there exists a solution $x(t)$, $\forall t \geq 0$ satisfying

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\Big(\sup_{\tau\in[0,\infty)}\{\|u_\tau\|\}\Big),$$

for all admissible inputs $u(t)$ and for all initial values $x(0)$, with β and γ being a $\mathcal{KL}$ and $\mathcal{K}_\infty$ function, respectively.

Definition 2 (ISS Lyapunov Function):
A continuously differentiable function $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$ is an input-to-state (ISS) Lyapunov function for (1) if there exists class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a class-$\mathcal{K}$ function γ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ satisfying the following conditions:

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq \alpha_2(\|x(t)\|), \tag{2}$$

$$\dot{V}(t) \leq -\alpha_3(\|x(t)\|) + \gamma(\|u(t)\|). \tag{3}$$

II. Problem Formulation and Preliminaries

This section presents the problem and briefly describes some preliminaries used in the next sections.

A. Problems description

This paper considers a feedback control strategy for networked control systems in the presence of bandwidth constraints in feedback path. To deal with channel constraints in the feedback loop, we formulate a novel robust event-triggered control algorithm for a class of nonlinear systems. Two control problems are solved for two different closed-loop architectures. In **Problem 1**, we design the robust controller and event-triggering law using an emulation-based approach in which controller and triggering law are designed separately.

Then, a co-design problem (**Problem 2**) is formulated to jointly design triggering law and controller.

1) Problem 1: Figure 1 shows the block diagram of the proposed robust control technique for Problem 1. In this diagram, the following elements are clearly appearing: (i) system, (ii) controller, and (iii) a communication network interconnecting system and controller. The states of the system are measured continuously by sensors at the system end. Information from sensors are shared with the controller through a communication network. In between sensor and controller, an event-monitoring unit continuously tracks an event condition. Specifically, when a predefined triggering event occurs, the monitoring unit ensures the proper transmission of the state variable to the controller. This robust control problem is addressed by means of an equivalent optimal control strategy based on the linear nominal model or a virtual dynamics of the original nonlinear system. The gain K of the controller and aperiodic state datum, $x(t_k)$, which is obtained from the nonlinear system serves to compute this event-triggering control rule $u(t_k) = Kx(t_k)$ stabilizing the closed-loop system in the presence of uncertainty. The input function is actuated aperiodically at instants $t_0, t_1, \cdots, t_k$, where t_k represents the latest such event. A zero-order hold (ZOH) at the actuator end holds the most recent actuated input data until a subsequent triggering event leads to the transmission of another input data. Here, the actuator is assumed to be embedded within the system, with an instantaneous update of the control input at the time of transmission. The primary concern of this paper is to propose a robust event-triggering rule that can withstand the system's nonlinearities and model uncertainty for a class of nonlinear systems.

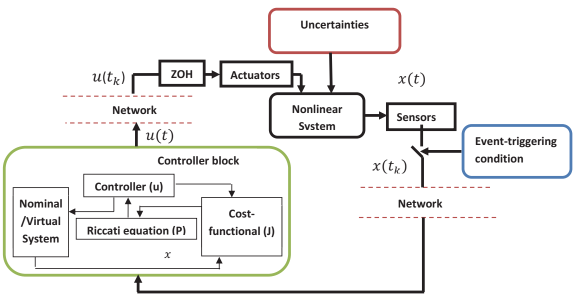

Fig. 1. Architecture I: Schematic block diagram of the developed event-triggered robust control strategy.

System description: Consider a state-space representation of the following class of nonlinear systems

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + B(u_1 + h(x)u_1) + Bf(x), \qquad (4)$$

where A and B are system matrices. The notations x and u_1 represent the state vector and control input for (4) respectively. The two nonlinear functions $h(x)$ and $f(x)$ are treated as uncertainty sources. Specifically, $h(x)$ corresponds to the uncertainty at the input level, while $f(x)$ embodies the uncertainty at the system's level. These functions satisfy the following assumptions:

Assumption 1: There exists a positive definite matrix F_1 that satisfies

$$f(x)^T f(x) \leq x^T F_1 x. \qquad (5)$$

Assumption 2: There exists a known function $h_{\max}(x)$ such that for all x, the function $h(x)$ satisfies

$$0 \leq h(x) \leq h_{\max}(x). \qquad (6)$$

Remark 1: For a given $f(x)$ in (5), $\|f(x)\|^2$ may not be quadratically bounded. In many cases, we can find the largest physically feasible region of x and determine a quadratic bound for $\|f(x)\|^2$. Assume such a bound is given by $f(x)^T f(x) \leq x^T F_1 x$ for some positive definite matrix F_1. Assumption 2 helps to prove that $h(x) \geq 0$. In [13], such Assumptions (1 and 2) are used to compute the upper bound of uncertainties.

In general, uncertainty in system dynamics is either **matched** or **mismatched**. From (4), it appears that this problem is afflicted by matched uncertainty since both $f(x)$ and $h(x)$ are associated with the nominal input matrix B. This means that the unknown part of (4) can be represented by a vector $\Delta(x, u_1)$ defined as

$$\Delta(x, u_1) = Bf(x) + Bh(x)u_1. \qquad (7)$$

From [4], the closed-loop system (4) with event-triggered control input $u_1(t_k)$ can be written as

$$\dot{x}(t) = Ax + B\{u_1(t_k) + h(x)u_1(t_k)\} + Bf(x), \qquad (8)$$

$$u_1(t_k) = K_1 x(t_k), \qquad (9)$$

where K_1 is the controller gain and $x(t_k)$ is the state information of (8) at the k^{th} event-triggering instant. To tackle aperiodic information $x(t_k)$, we introduce an error variable, $e(t)$, defined as

$$e(t) = x(t_k) - x(t), \quad t \in [t_k, \ t_{k+1}). \qquad (10)$$

We remark that (4) encompasses a fairly rich class of nonlinear problems. For instance, any problem in the form of Euler–Lagrange can be expressed as (4).

Example 1: The Euler–Lagrange (EL) system dynamics is governed by

$$M(q)\ddot{q} + N(q, \dot{q}) = \tau, \qquad (11)$$

where $N(q, \dot{q}) = V(q, \dot{q}) + F(\dot{q}) + G(q)$. The vectors $q \in \mathbb{R}^n$ and $\tau \in \mathbb{R}^n$ denote the state variables and generalized forces, respectively. The inertia matrix, Coriolis vector, gravity vector and friction vector are also denoted by $M(q) \in \mathbb{R}^{n\times n}$, $V(q, \dot{q})$, $G(q)$ and $F(\dot{q}) \in \mathbb{R}^n$, respectively. In EL system, Assumptions 1 and 2 would translate into the following two conditions.

Condition 1: There exist positive matrices $M_0(q)$ and $M_{\min}(q)$ such that the following inequalities hold

$$M_{\min}(q) \leq M(q) \leq M_0(q).$$

Condition 2: There exists a function $n_{\max}(q, \dot{q})$ and vector $N_0(q, \dot{q})$ such that the following inequality holds

$$\|N(q, \dot{q}) - N_0(q, \dot{q})\| \leq n_{\max}(q, \dot{q}).$$

As a result of uncertain load variations and unmodeled dissipative effects, the terms $M(q)$ and $N(q, \dot{q})$ in (11) carry some levels of uncertainty. With uncertainty accounted for and letting the state vector $x = [q, \dot{q}]^T$, the state-space representation reads as (4) with $A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$. For EL systems, the two nonlinearity sources at the input and system level are given by

$$h(x) = M(q)^{-1} M_0(q) - I \geq 0, \tag{12}$$

$$f(x) = M(q)^{-1}(N_0(q, \dot{q}) - N(q, \dot{q})). \tag{13}$$

□

To regulate the closed-loop behavior of (8) with limited control actions, the following problem is formulated.

$\mathbf{P}_1$– Problem Statement: Design the robust event-triggered state feedback control law (9) to regulate the closed-loop behavior of the event-triggered system (8) such that it is input-to-state stable with respect to its measurement error $e(t)$ in the presence of uncertainty (7).

2) Problem 2: In **Problem 1**, a robust control problem has been considered for a class of nonlinear systems in the presence of communication constraints. In **Problem 2**, we consider both communication cost and system uncertainty, and propose an optimal control framework jointly optimizing the two costs—communication cost and the cost associated with system uncertainty.

To derive the results, a finite-horizon optimal control problem for linear systems is proposed. Such a finite-horizon control is considered as it constitutes a more realistic scenario in practical problems. In addition, the approach taken in **Problem 1** considers a zero-order hold (ZOH) at the actuator end, such that the last transmitted state and control input are held constant until new information is transmitted (see Fig. 1). Therefore, for **Problem 2** we consider a ZOH-free robust control technique with optimal event-triggered feedback. The block diagram of the proposed control technique is shown in Fig. 2. The state of the uncertain system is measured by sensors and each sensor has a copy of the nominal model as originally proposed in [18], [19]. The presence of nominal model at sensor end helps to compute the error between actual state $x(t)$ and nominal state $x_n(t)$:

$$\hat{e}(t) = x_n(t) - x(t). \tag{14}$$

The variable $\hat{e}(t)$ measures the deviation of actual closed-loop performance form the nominal behavior of the system. The event-triggering unit computes $\hat{e}(t)$[1] and solves an optimization problem considering the communication cost to obtain the optimal transmission sequence. Based on the obtained optimal transmission sequence, the actual state is transferred through the communication channel. A dynamic programming based technique is used to solve the optimization problem. In the previous approach, the triggering condition depends on the growth of $e(t) = x(t_k) - x(t)$. Here, the time instant t_k represents the event-triggering instants as mentioned in **Problem 1**. The measurement transmitted to the controller end remains unchanged until new information is received. Yet, here, the nominal model is available at the controller end to estimate the nominal behavior of the system. At the event-triggering instant t_k, the state of the nominal model within the controller is replaced by the new measurement $x(t)$ available from the original uncertain system. The nominal system state is used to compute the control law $u_2(t) = K_2 x_n(t)$, where K_2 is the controller gain. Hence, within two consecutive event-triggering instants, the control input is generated by using the nominal model state

$$\dot{x}_n(t) = (A + BK_2)x_n(t), \ \forall \, t \in \ [t_k, \ t_{k+1}). \tag{15}$$

Applying the control input u_2 in (4) leads to

$$\dot{x}(t) = Ax + B\{u_2(t) + h(x)u_2(t)\} + Bf(x), \tag{16}$$

$$u_2(t) = K_2 x_n(t) = K_2(x(t) + \hat{e}(t)), \tag{17}$$

with $\hat{e}(t)$ defined in (14). In (15), at every event-triggering instant t_k, the nominal state $x_n(t)$ is replaced by the original state $x(t)$ and it resets the error $\hat{e}(t)$ to zero. To describe the network constraints, consider a variable δ_t which decides whether the state information is transmitted or not. The variable δ_t is defined as

$$\delta_t = \begin{cases} 1 & \text{when } x(t) \text{ is transmitted,} \\ 0 & \text{if no information transmitted.} \end{cases} \tag{18}$$

The switch of this binary decision variable from 0 to 1 depends on the selection of a particular event-triggering law. Let Ξ be a triggering law whose evolution depends on the error variable $\hat{e}(t)$. The design objective is to define the robust controller

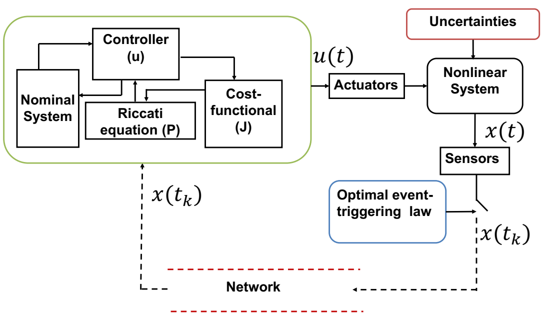

Fig. 2. Architecture II: Block diagram of proposed finite-horizon robust control technique with event-triggered feedback.

[1]Here we have used two error variables as $e(t)$ and $\hat{e}(t)$. The variable $e(t)$ is used to compute the difference between last transmitted state $x(t_k)$ and current state $x(t)$ that is $e(t) = x(t_k) - x(t)$ where $t \in [t_k, t_{k+1})$. On the other hand, $\hat{e}(t)$ measures the difference between nominal state $x_n(t)$ and the state of uncertain system $x(t)$, that means $\hat{e}(t) = x_n(t) - x(t)$.

K_2 and the event-triggering law Ξ that minimizes a certain cost-functional. Here, we consider the cost-functional

$$J_2 = \int_0^T (x^T Q x + x^T F_1 x + u_2^T u_2 + \lambda \delta_t) dt, \quad (19)$$

where $\lambda > 0$ is a penalty due to any exchange of information between sensor, controller and actuator over the transmission network and T denotes the final time of execution. To regulate the state of (16) by event-triggered feedback with the transmission cost $\int_0^T \lambda \delta_t dt$, the following problem is introduced.

Problem statement ($\mathbf{P}_2$): Design a finite-horizon, linear, robust state feedback control law $u_2(t) = K_2 x_n(t)$ and an optimal event-triggering law $\Xi^*(\hat{e}(t))$ for (16) that ensures stability in the presence of uncertainty (7).

Proposed solution: To solve problems $\mathbf{P_1}$ and $\mathbf{P_2}$, two different steps are adopted. First, results from optimal control theory are used to develop a robust control strategy. As a next step, an event-triggering criterion is established to ensure input-to-state stability of (8) or (16). This criterion is obtained from the assumption of the existence of an ISS-stable Lyapunov function $V(x) = x^T P x, \quad P \geq 0$.

B. Controller design

To determine the state feedback gains for Problems $\mathbf{P_1}$ and $\mathbf{P_2}$, this paper adopts both emulation and co-design approaches. Within the emulation approach, initially, the gain matrices are derived assuming that feedback information is continuously available. Next, some techniques are developed to consider network effects. Within the co-design approach, first, a robust controller gain is designed for (16), and subsequently, an optimal event-triggering law is introduced to reduce the number of data transmissions over the network. The controller design process is discussed next.

Aim: Design state feedback controllers K_1 and K_2 such that system (4) and (16) remain stable in the presence of bounded uncertainty (7).

To solve this robust control problem, an optimal control approach using a robust control algorithm is adopted. The main idea is to design the optimal control input for the linear nominal system that minimizes a modified cost function. The term "modified" is used here to characterize the cost function given its dependence on the maximum variation (i.e. upper bound) of uncertainty. Then, it is shown that this derived optimal input is also a robust solution to the original system in the presence of uncertainty. For the uncertain system (4), its corresponding nominal system and cost function are derived in what follows.

- **For Problem $\mathbf{P_1}$:** The nominal dynamical law for system (4) in the presence of uncertainty becomes

  $$\dot{x} = Ax + Bu_1, \quad (20)$$

  and the modified cost function for this matched uncertain system (4) is given by

  $$J_1 = \int_0^\infty (x^T F_1 x + x^T Q x + u_1^T u_1) dt, \quad (21)$$

  with $Q \geq 0$. The matrix $F_1 \geq 0$ is the upper bound of the uncertainty defined in (5).
- **For Problem $\mathbf{P_2}$:** To design the robust controller gain for (16), we adopt the optimal control framework.

To obtain a robust controller in this optimal control approach, we use the following Lemma stated in [11], [12].

Lemma 1: The optimal control solutions for nominal systems (20) and (15) with modified cost functions (21) and (19) respectively is robust for the original systems (4) and (16) in the presence of all bounded variations of uncertainty (7).

A proof for Lemma 1 can be found in [11], [12].

Based on Lemma 1, the robust controller gain matrices can be obtained by solving the LQR problem. According to optimal control theory [17], the optimal control signal for (20) minimizing the cost function (21) is

$$u_1 = -\underbrace{B^T P_1}_{K_1} x, \quad (22)$$

where P_1 satisfies the following Riccati equation

$$P_1 A + A^T P_1 - P_1 B B^T P_1 + F_1 + Q = 0. \quad (23)$$

Similarly, for system (15), we consider the optimal input

$$u_2 = \underbrace{-B^T P(t)}_{K_2(t)} x_n(t), \quad (24)$$

where $P(t)$ is the solution of the following differential Riccati equation (DRE)

$$-\dot{P} = A^T P + PA - PBB^T P + Q + F_1. \quad (25)$$

For simplicity of notation, in what follows, we omit the argument t in $P(t)$. The steps to obtain the numerical solution of (25) are discussed in [21].

Here K_1 and K_2 are gain matrices. The aperiodic state information $x(t_k)$, $x_n(t)$ and controller gain matrices are used, respectively to derive the event-triggered control law, which is discussed next.

III. Main results

This section presents event-triggering conditions and stability results of (8) and (16), in the presence of uncertainty (7). The solution of problems P_1 is described in the following Theorem.

Theorem 1: Let $\sigma \in (0,1)$ and the optimal controller gain K_1 derived for the nominal system (20) with cost function (21). The event-triggered control law (9) ensures ISS of the uncertain system (8) if the control input actuation instant satisfies the following sequence

$$t_0 = 0, \ t_{k+1} = \inf\{t \in \mathbb{R} | t > t_k \wedge \mu_1 \|x\|^2 - \|e\|^2 \leq 0\}, \quad (26)$$

where the variable μ_1 is defined as

$$\mu_1 = \frac{\sigma \lambda_{\min}^2(Q)}{8(1 + \|h_{\max}\|^2)\|K_1^T K_1\|^2}. \quad (27)$$

Proof: To prove the ISS-stability of uncertain system (8) with control input (9), it is necessary to reformulate $\dot{V}(x)$ such that it satisfies (3). Consider the Lyapunov function for (8) in the form of a positive smooth function $V(x) = x^T P_1 x$. To ensure the stability of (8), $\dot{V}(x)$ is recast as

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)^T (Ax + B\{(K_1 x + K_1 e) + h(x)(K_1 x + K_1 e)\} + Bf(x)). \tag{28}$$

The function $V(x)$ is a Lyapunov function for (20) that satisfies the Hamilton–Jacobi–Bellman (HJB) equation

$$\min_{u_1} \left(x^T F_1 x + x^T Q x + u_1^T u_1 + V_x^T (Ax + Bu_1)\right) = 0, \tag{29}$$

where matrix V_x denotes $\frac{\partial V}{\partial x}$. For a selection of Lyapunov function $V(x) = x^T P_1 x$, the HJB equation (29) reduces to a Riccati equation (23). The optimal input u_1 must satisfy (29); that means

$$x^T F_1 x + x^T Q x + u_1^T u_1 + V_x^T (Ax + Bu_1) = 0, \tag{30}$$

$$2u_1^T = -V_x^T B. \tag{31}$$

Using (30) and (31), Eq. (28) is simplified as

$$\begin{aligned}\dot{V}(x) &\leq -x^T F_1 x + f(x)^T f(x) - x^T Q x - 2u_1^T K_1 e \\ &- 2u_1^T h u_1 - 2u_1^T h K_1 e - (u_1 + f(x))^T (u_1 + f(x)) \\ &\leq -\frac{\lambda_{\min}(Q)}{2}\|x\|^2 + \frac{4}{\lambda_{\min}(Q)} \\ &\left(\|K_1^T K_1\| + \|K_1^T K_1\|^2 \|h_{\max}\|^2\right) \|e\|^2\end{aligned} \tag{32}$$

The inequality (32) ensures the ISS of (8) with respect to measurement error e. From (3) and (32), it is observed that the actuation of control input is solely required upon violation of the event-triggering criterion (26). The procedure to realize the control law designed for **Problem 1** is presented in Algorithm 1; see Appendix A.

Solution of $\mathbf{P_2}$ (Optimal event-triggering law):
From the event-triggering law $\Xi(\hat{e}(t))$, it can be stated that the variable $\hat{e}(t)$ influences the number of transmissions over the network. In order to design the optimal event-triggering law, it is necessary to define the dynamics of $\hat{e}(t)$. Using (15) and (16), $\hat{e}(t)$ evolves based on the following dynamics

$$\dot{\hat{e}}(t) = A(x_n(t) - x(t)) - Bf(x) - Bh(x)u_2.$$

Neglecting the uncertain terms $f(x)$ and $h(x)$, the nominal error dynamics reads

$$\dot{\hat{e}}(t) = A\hat{e}(t), \ \forall\ t \in [t_k,\ t_{k+1}). \tag{33}$$

At event-triggering instant t_k, $\hat{e}(t)$ is zero as the nominal state $x_n(t)$ is replaced by actual state $x(t)$. To obtain the optimal event-triggering, the following optimization problem is solved:

$$\delta_t^* = \arg\min_{\delta_t} J(\hat{e}(t), \delta_t) = \int_0^T \{(1 - d_t)\hat{e}^T K_2^T K_2 \hat{e} + \lambda \delta_t\} dt, \tag{34}$$

$$\text{subject to: (33) and } \hat{e}(t) \in \Omega, \tag{35}$$

where

$$\Omega = \{\hat{e}(t) \in \mathbb{R}^n |\ \|\hat{e}(t)\|^2 \leq \xi\}. \tag{36}$$

The scalar $\xi > 0$ is computed from the stability results.

Remark 2: The term $u_2^T(t)u_2(t)$ in (19) can be rewritten as $(K_2 x + K_2\hat{e}(t))^T (K_2 x + K_2 \hat{e}(t))$ using (14). This helps to rewrite the cost-functional (19). To compute the optimal controller $u_2(t)$ for the nominal system, the terms δ_t and $\hat{e}(t)$ can be neglected from the minimization, since δ_t is constant and controller gain design is independent of error $\hat{e}(t)$. However, the triggering condition design depends on the variable δ_t and $\hat{e}(t)$.

To obtain the robust controller and optimal event-triggering law, the following Theorem is proposed

Theorem 2: The optimal state feedback gain K_2 derived in (24) robustly stabilizes (16) in the presence of uncertainty (7) if control inputs are actuated based on the optimal event triggering sequence δ_t^* obtained from (34) and (35).

Proof: Consider the Lyapunov function $V(x) = x^T P(t) x$. Then $\dot{V}$ is computed as

$$\begin{aligned}\dot{V}(x) &= x^T(A^T P + PA + \dot{P})x - 2u_2^T u_2 - u_2^T h^T u_2 \\ &-u_2^T h u_2 - f(x)^T u_2 - u_2^T f(x) + \hat{e}^T K_2^T B^T P x \\ &+x^T P B K_2 \hat{e} + x^T P B h K_2 \hat{e} + \hat{e}^T K_2^T h^T B^T P x.\end{aligned}$$

Using (24) and (25), the above equality is simplified as

$$\begin{aligned}\dot{V}(x) \quad \leq \quad & -x^T Q x - (x^T F_1 x - f(x)^T f(x)) \\ & -(u_2 + f(x))^T (u_2 + f(x)) \\ & +\hat{e}^T K_2^T K_2 x + x^T K_2^T h K_2 \hat{e} \\ & +x^T K_2^T K_2 \hat{e} + \hat{e}^T K_2^T h^T K_2 x.\end{aligned} \tag{37}$$

Applying (5) and (6), the inequality (37) reduces to

$$\begin{aligned}\dot{V}(x) \quad \leq \quad & -\frac{\lambda_{\min}(Q)}{2}\|x\|^2 + \frac{4}{\lambda_{\min}(Q)}\left(\left\|K_2^T K_2\right\|^2\right. \\ & \left. + \left\|K_2^T K_2\right\|^2 \|h_{\max}\|^2\right) \|\hat{e}\|^2.\end{aligned} \tag{38}$$

This ensures that the closed-loop system (16) is ISS with the event-triggering law Ξ^*. The threshold ξ in (36) can be computed from (38) as

$$\xi \leq \mu_2 \|x\|^2. \tag{39}$$

where $\mu_2 = \frac{\sigma \lambda_{\min}^2(Q)}{8(1+\|h_{\max}\|^2)\left\|K_2^T K_2\right\|^2}$ and $\sigma \in (0,\ 1)$. The steps to realize the robust control law for (16) with optimal event-triggering law $\Xi^*(\hat{e}(t))$ are detailed in Algorithm 2 (see Appendix A).

Remark 3: Although μ_1 and μ_2 appear identical—implying that event-triggering conditions are the same for both problems, they actually correspond to different conditions since the controller gains K_1 and K_2, Riccati solutions P_1 and P and the error variables e and $\hat{e}$ are not identical.

IV. Simulation Results

This section tests the theoretical results derived in Section III for a class of nonlinear system. Let us consider system (4) with state and system matrices given by $x = [x_1\ x_2]^T$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

TABLE I
EVENT-TRIGGERED CONTROL VS. CONTINUOUS CONTROL

Control Strategy	τ_{max} (sec)	τ_{min} (sec)	u_{total}
Continuous control	0.008	0.008	500
Event-triggered control	0.27	0.008	316

The nonlinearities correspond to $h(x) = \frac{2w_1 x_1^2}{(x_1^2+1)}$ and $f(x) = 2w_2 x_1 \sin^2(x_1)\cos(x_2)$, with w_1 and w_2 being scalar parameters which are uncertain and can vary in the interval $[0, 1]$. The upper bound of h_{max} is considered as $\|h_{\text{max}}\| = 2$. The controller gain is computed using (22) which minimizes (21). We consider the matrices $F_1 = 4I$ and $Q = 10I$. To compute K_1, the Riccati equation (23) is solved. The positive definite solution P_1 of (23) is used to compute the optimal input

$$u = -\begin{bmatrix}10 & 10.4\end{bmatrix} x.$$

To realize the event-triggering sequence (26), the design parameter σ is selected to be 0.6. The numerical simulation runs for 4 time units with the initial condition $[0.1, -0.1]^T$.

For all simulations, we extracted 100 random samples of w_1 and w_2 within the interval $[0, 1]$ and tested the performance of the designed controller. Figure IV shows the convergence of state trajectories for different values of w_1 and w_2. As it can be seen from Fig. IV, all states converge to zero for various samples extracted from the set of uncertainty which confirms the robustness of the designed controller. Figure IV shows the inter-event time of execution instants, and reveals that the number of computed control inputs is drastically reduced, thereby confirming the reduction in the ensuing communication cost. A comparative study with the conventional continuous control approach is shown in Table I. It confirms that the total number of actuations u_{total} for the event-triggered case is far less than that of the continuous control technique. The symbols τ_{max} and τ_{min} denote the maximum and minimum inter-event time—the time between two consecutive events—of event generation.

To realize the optimal event-triggered control approach proposed in Section III, we consider the same example discussed above. The control law is computed for a finite-horizon $T = 4$ seconds. The control law (17) is computed numerically using the solution of DRE (25). To obtain the optimal event-triggering law Ξ^*, the dynamic programming based optimization problem is formulated which generates the optimal triggering instants δ_t^*. Sensors at the system end transmit state x based on δ_t^*. The convergence of states when the optimal triggering law Ξ^* has been used, is shown in Fig. IV. The scalar λ is selected to be 0.4. Figure IV shows the evolution of the switching variable δ_t^* for a given run-time. Table II compares the total number of transmission between event-triggered control technique with optimal triggering and the conventional continuous approach. Again, we observe that the total number of transmissions is significantly reduced thereby confirming the efficacy of the proposed approach.

TABLE II
COMPARISON OF ROBUST EVENT-TRIGGERED CONTROL WITH OPTIMAL TRIGGERING VS. CONTINUOUS CONTROL

Control Strategy	τ_{max}(sec.)	τ_{min}(sec.)	u_{total}
Continuous control	0.04	0.04	100
Finite-horizon event-triggered control	1.8	0.04	36

V. CONCLUSIONS

In this paper, we present a robust event-triggered controller for a class of nonlinear systems. The nonlinearity in the system dynamics and parameter variations are considered as a source of uncertainty. To stabilize such systems, the linear part of the system model is considered to design the controller gain. Then, a linear robust control law is derived to stabilize such systems with aperiodic feedback information. The robust control law is derived adopting the optimal control framework with both infinite and finite-horizon costs. To deal with aperiodicity of feedback information, this paper introduces two different triggering laws. The triggering condition and stability results are derived based on the Input-to-State Stability theory. The derived triggering laws optimally manage the network bandwidth in the presence of uncertainty. The effectiveness of the proposed control laws is illustrated through numerical simulations. The numerical results confirms the effectiveness of the developed control strategies against the classical continuous feedback control approach.

APPENDIX

Algorithm 1 Robust Event-Triggered Control for Problem $\mathbf{P}_1$

1: Initialization: $x \leftarrow x(0)$, $x(t_k) \leftarrow x(0)$.
2: Using A, B, F_1, σ, ϵ, η compute K_1 from (22).
3: Compute $\|x(t)\|$, $\|e(t)\|$ and μ_1 using (27).
4: **if** $\|e\|^2 \geq \mu_1 \|x\|^2$ **then**
5:     Send $x(t_k)$ from sensor to controller.
6:     Compute and update the control laws (9)—for the system (8).
7: **else**
8:     Hold the previous input
9: **end if**
10: Return to line 3

REFERENCES

[1] L. Zhang and D. H. Varsakelis, "LQG control under limited communication", *44th IEEE Conf. on Decision and Control*, pp. 185-190, 2005.

[2] K. Astrom and B. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems", *41st IEEE Conf. on Decision and Control*, pp. 2011-2016, 2002.

[3] W.P.M.H. Heemels, K.H. Johansson and P. Tabuada, "An introduction to event-triggered and self-triggered control", *51st IEEE Conf. on Decision and Control*, pp. 3270-3285, 2012.

[4] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks", *IEEE Trans. on Autom. Control*, vol. 52, no. 9, pp. 1680-1685, 2007.

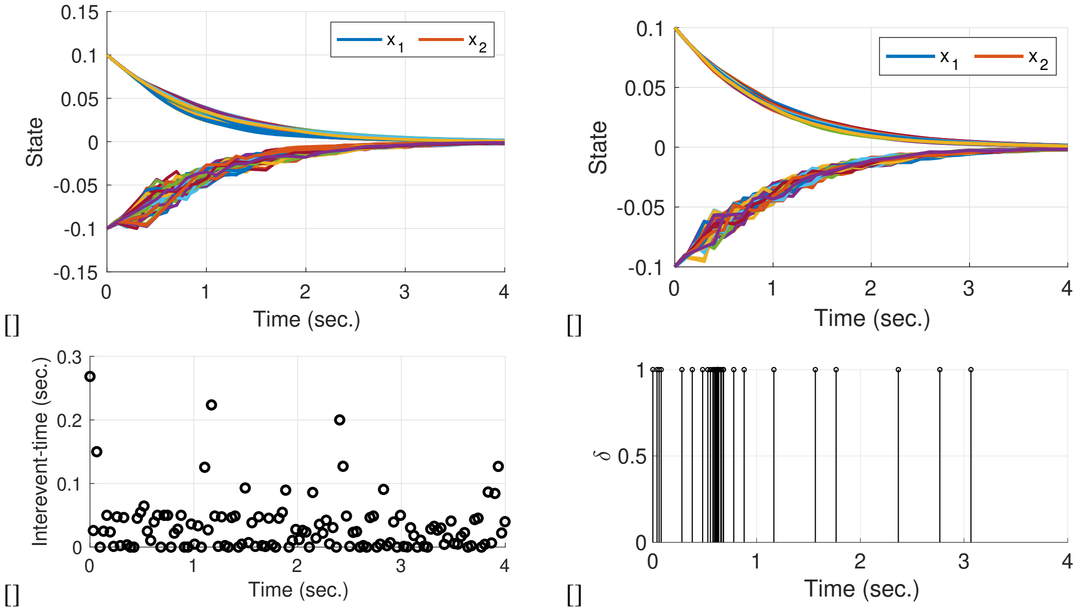

Fig. 3. (a): Stabilization of states for 100 random uncertain samples of w_1 and w_2. (b): Convergence of states with uncertainty for 100 random uncertain samples of w_1 and w_2 using optimal event-triggered control for $T = 4$ sec. (c): Inter-event time for $w_1 = 0.5,\ w_2 = 0.5$. (d): Evolution of δ with time for $w_1 = 0.3$ and $w_2 = 0.3$.

Algorithm 2 Robust Event-Triggered Control with Optimal Triggering (Problem $\mathbf{P_2}$)

1: Initialization: $x \leftarrow x(0)$, $t \leftarrow 0$.
2: Given: A, B, F_1, T,
3: Compute K_2 using (24), (25).
4: Compute $\|\hat{e}(t)\|$, ξ using (33) and (39) and solve optimization problem (34) and (35) to obtain δ_t^*.
5: **if** $\delta_t^* = 1$ **then**
6: Send $x(t)$ from sensor to controller.
7: Replace $x_n(t)$ with $x(t)$ in (15).
8: Compute and update the control laws (24) using (15)—for the system (16).
9: **else**
10: Compute and update the control laws (24) using (15) —for the system (16).
11: **end if**
12: Return to line 3

[5] S. Trimpe and R. DAndrea, "Event-based state estimation with variance-based triggering", *51st IEEE Conf. on Decision and Control*, pp. 6583-6590, 2012.

[6] P. Tallapragada and N. Chopra, "On event triggered tracking for non-linear systems", *IEEE Trans. on Autom. Control*, vol. 58, no. 9, pp. 2343-2348, 2013.

[7] W. P. M. H. Heemels, M. C. F. Donkers, and A. R. Tell,""Periodic event-triggered control for linear systems,"" *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 847–861, 2013.

[8] X. Wang and M. D. Lemmon, "Self-triggered feedback control systems with finite-gain stability", *IEEE Trans. Autom. Control*, vol. 54, no.3, pp. 452-462, 2009.

[9] N. S. Tripathy, I. N. Kar, and K. Paul, "Stabilization of uncertain discrete-time linear system with limited communication", *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4727-4733, 2017.

[10] N. S. Tripathy, I. N. Kar, and K. Paul, "Model Based Robust Control Law for Linear Event-triggered System", *Asian Journal of Control*, vol. 18, no. 5, pp. 1-6, 2016.

[11] F. Lin and R. D. Brandt, An optimal control approach to robust control of robot manipulators, *IEEE Trans. Robot. Automat.*, vol. 14, no. 1, pp. 69-77, 1998.

[12] F. Lin, "An optimal control approach to robust control design", *Int. J. Control* vol. 73, no.3, pp. 177-186, 2000.

[13] H. Pan and M. Xin, "Nonlinear robust and optimal control of robot manipulators", *Nonlinear Dynamics*, vol. 76 , no. 1, pp. 237-254, 2014.

[14] E. D. Sontag, Input to State Stability: Basic Concepts and Results, *Nonlinear and Optimal Control Theory*, pp. 163-220, 2008.

[15] I. R. Petersen, "Structural stabilization of uncertain systems: Necessity of the matching condition", *SIAM J. Control Optim.*, vol. 23, no. 2, pp. 286-296, 1985.

[16] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, 3rd Edition, New Jersey, 2002.

[17] D. S. Naidu, *Optimal Control Systems*. CRC press, India, 2009.

[18] L. A. Montestruqu and P. J. Antsaklis, "On the model-based control of networked systems", *Automatica*, vol. 39, no.10, pp. 1837-1843, 2003.

[19] E. Garcia and P. J. Antsaklis, "Model-based event-triggered control for systems with quantization and time-varying network delays", *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 422-434, 2013.

[20] H. J. Kushner and P. Dupuis, "Numerical Methods for Stochastic Control Problems in Continuous Time", *Springer*, 2nd ed., 2001.

[21] J. Nazarzadeh, M. Razzaghi and K.Y. Nikravesh, "Solution of the matrix Riccati equation for the linear quadratic control problems", *Mathl. Comput. Modelling*, vol. 27, no.7, pp. 51-55, 1998.