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Finite-time Event-triggered Control for a Class of Nonlinear Systems

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Abstract— **This paper considers an approximate solution of the event-triggered Hamilton-Jacobi-Bellman (ET-HJB) equation to derive a finite-time suboptimal event-triggered control law for a class of nonlinear systems. To reduce the communication and computation overhead, the control law is computed and actuated after violating a predefined state-dependent event triggering condition. To obtain the controller gain, the ET-HJB equation is approximated as a state-dependent differential Riccati equation (SDRE). After converting the ET-HJB into an SDRE, a frozen time concept is used to eliminate the issues related to state dependency in the system and input matrices between two consecutive events. This helps reframe the SDRE into a simple differential Riccati equation (DRE), where the state-dependent system and input matrices remain fixed until the next event occurs. Using the solution of a differential Lyapunov equation, the solution of the DRE is computed forward in time. The designed event-triggered control law is readily amenable to an online implementation, and also it ensures the input-to-state stability of closed-loop systems. Simulation results are reported to prove the efficacy of the proposed control approach.**

I. Introduction

Generally, in Cyber-Physical Systems (CPS), each physical system shares its local information with the other subsystems through a digital network. Due to the shared nature of the communicating channel, controlling such systems with continuous or periodic control laws requires significant bandwidth requirements [1]-[2]. Recently, event-triggered-based control techniques have been proposed in [3]-[8] to reduce the information requirements for realizing a stable control law. In event-triggered control, sensing at the system-end and actuation at the controller-end happens only when a prespecified event condition is violated. This so-called event triggering condition mostly depends on the system's current states or outputs. The primary shortcoming of continuous-time event-triggered control is that it requires continuous monitoring of the event condition. In [7]-[8], Heemels et al. proposed an event-triggering technique where the event-condition is monitored periodically. To avoid continuous or periodic monitoring, self-triggered control techniques have been reported in [9]-[10], where the next event occurring instant is computed analytically using the system's state at the previous sampling instant. Maximizing the inter-event time is the key aim of the event-triggered or self-triggered control to reduce the total communication requirements further. The efficacy of aperiodic sensing and actuation over the continuous or periodic one in the context of CPS are expressed in [1]-[2].

Controlling nonlinear systems with a finite-time event-triggered optimal control law is a challenging control problem due to its time dependency of the solution. Few attempts have been made in the past to compute the optimal solution for a linear event-triggered system with a quadratic cost functional [11], [12] and [13]. However, extending these results to nonlinear systems with finite-time convergence is not straightforward. In general, to design a finite-time optimal control law for a nonlinear system, it is essential to solve a continuous-time Hamilton-Jacobi-Bellman (CT-HJB) equation. Nevertheless, to compute optimal event-triggered control input, it is necessary to convert the CT-HJB equation into an event-triggered HJB (ET-HJB) [16]. Solving the HJB equation is computationally intensive as it is a partial differential equation. Different techniques like neural networks [14]-[17], dynamic programming [13], [11] have been considered to approximate both the CT-HJB and the ET-HJB equation. However, these approximation techniques are also computationally taxing. Therefore, computationally complex approximation technique to realize a control law is difficult to implement in resource-constrained CPS. To address this critical issue, this paper proposes a procedure to convert the ET-HJB equation into a Riccati equation for a class of nonlinear systems, inspired by the results reported in [18]. In [18], the conversion processes from a CT-HJB to an SDRE have been discussed. However, in this paper, instead of the CT-HJB, the ET-HJB is considered and converted into an SDRE which is a challenging task in itself as the control inputs are aperiodic in nature.

Here, an attempt is made to realize an event-triggered control law for a class of nonlinear systems whose system dynamics can be represented in a state-dependent coefficient (SDC) form [19]. With this SDC form, the state and input functions are converted to a linear-like structure. The SDC representation of nonlinear systems is applicable to several important practical systems like under-actuated robot [27]-[28], missile [29], spacecraft [30], and level control of tank systems [31]. Figure 1 depicts the overall block diagram of the proposed control method. Here, the system and controller are connected using a communication network. The state and input information are transmitted aperiodically to the controller and actuator, respectively. To design a finite-time event-triggered control law for controlling such systems, an ET-HJB equation is considered. After some

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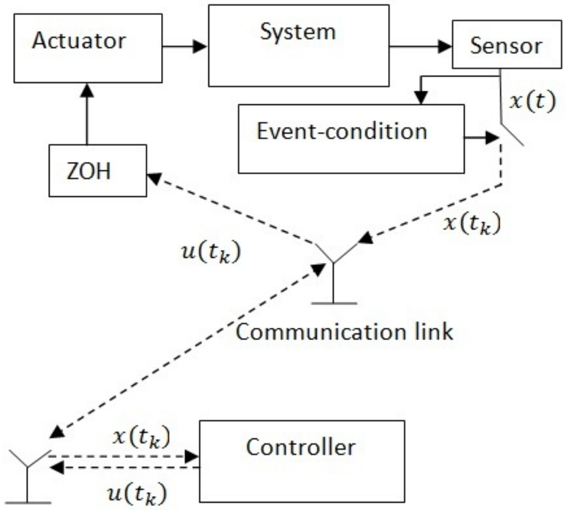


Fig. 1: Block diagram of proposed event-triggered control technique. The dotted lines are used to represent the aperiodic transmission of data over the communication channel. The symbols x and u denote the system state and control input, respectively.

mathematical simplification, the ET-HJB is converted into an SDRE. Applying the frozen-time concept within two consecutive events, the SDRE reduces to a simple differential Riccati equation (DRE). The analytical solution of DRE is calculated using the solution of a Differential Lyapunov Equation (DLE). In turn, the solution of the DLE is used to compute the control law at every event-triggering instant. This paper reports the following contributions:

(i) A proposed finite-time optimal control approach applicable to a class of nonlinear systems (whose system dynamics can be expressed in SDC form). To generate the control input, the ET-HJB is converted into an SDRE. The solution of SDRE is computed based on the solution of the DLE. Using the analytical solution of the SDRE for a final time t_f, the control input is computed at every event-triggering instants.

(ii) An event-triggering condition for a nonlinear systems is proposed by solving an SDRE equation. The event-triggered control law ensures the ISS of the closed-loop system. A comparative study between continuous and event-triggered method is carried out to highlight the contribution of this paper over the existing literature.

(iii) An example with simulation results is used to validate the proposed control algorithm.

Notation: The symbol $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. $\mathbb{R}^n$ represents the n dimensional Euclidean real space and $\mathbb{R}^{n\times m}$ is the set of all $(n \times m)$ real matrices. The notations $X \leq 0$, X^{-1} and X^T denote the negative semi-definiteness, inverse and transpose of matrix X, respectively. The notation I refers to the identity matrix. The minimum and maximum eigenvalue of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ are represented by the notations $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ respectively. A set Ω is used to denote a continuous Lipschitz compact set where state x (including the initial points) satisfy the condition $x \in \Omega$ [20]. A continuous function f: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is $\mathcal{K}_\infty$ if it is continuous and strictly increasing and it satisfies $f(0) = 0$ and $f(s) \to \infty$ as $s \to \infty$. The function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$, if it is strictly increasing and it satisfies $f(0) = 0$. A continuous function $\beta(r, s) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$ function, if it is class $\mathcal{K}$ function with respect to r for a fixed s and it is strictly decreasing with respect to s when r is fixed [20].

II. Problem formulation

Consider a continuous-time input-affine nonlinear system with state-dependent coefficients as [19]

$$\dot{x}(t) = A(x)x(t) + B(x)u(t), \tag{1}$$

where state and input matrices $A(x) \in \mathbb{R}^{n\times n}$ and $B(x) \in \mathbb{R}^{n\times m}$. Here $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the system state and input vector respectively. To stabilize (1), a finite-time optimal stabilizing control law is designed by minimizing the following quadratic cost-functional

$$J = \frac{1}{2}x_f^T F x_f + \frac{1}{2}\int_{t_0}^{t_f} \big(x^T Q x + u^T R u\big)dt, \tag{2}$$

where matrices $F \geq 0$, $Q \geq 0$ and $R > 0$. The time-instants t_0 and t_f in (2) denote the initial and final time respectively. To compute the optimal input for (1) with the cost-functional (2), the Hamiltonian H is defined as

$$H\left(x, u(t), \frac{\partial J}{\partial x}\right) = \frac{1}{2}\{x^T(t)Qx(t) + u^T(t)Ru(t)\} + \frac{\partial J}{\partial x}^T\Big\{A(x)x(t) + B(x)u(t)\Big\}. \tag{3}$$

Using optimal control theory results, the optimal input $u^*(t)$, should minimize H, that means

$$\frac{\partial H\big(x, u^*(t), \frac{\partial J^*(x,t)}{\partial x}\big)}{\partial u^*(t)} = 0. \tag{4}$$

The notation $J^*(x, t)$ denotes the optimal value of the cost-functional J. After simplification, (4) reduces to

$$u^*(t) = K(x) = -R^{-1}B(x)^T \frac{\partial J^*(x,t)}{\partial x}, \tag{5}$$

where $K(x)$ is a controller gain function. Also $u^*(t)$ satisfies the well-known HJB equation, i.e.

$$-\frac{\partial J^*(x,t)}{\partial t} = \frac{1}{2}\{x^T Q x + u^{*T} R u^*\} + \frac{\partial J^*(x,t)}{\partial x}^T\Big\{A(x)x(t) + B(x)u^*(t)\Big\}. \tag{6}$$

Now, to generate $u^*(t)$ from (5), the solution J^* of the HJB equation (6) is essential. To avoid the difficulty of solving (6), Heydari et al. [18] proposed an approximation technique to generate the optimal control input $u^*(t)$ by converting (6) into the following SDRE

$$\begin{aligned}\dot{P}(x,t) =& P(x,t)A(x) + A^T P(x,t) \\ & - P(x,t)B(x)R^{-1}B^T(x)P(x,t) + Q.\end{aligned} \tag{7}$$

The final boundary condition is

$$P(x,t_f) = F. \tag{8}$$

With the condition (8), the solution $P(x,t)$ of (7) is used to compute the approximate solution of (5) as

$$u(t) = -R^{-1}B(x)^T P(x,t)x(t). \tag{9}$$

The approximate input (9) makes the closed loop system (1) stable.

The realization of the control law (9) for a CPS or networked control systems (NCS) requires continuous transmission of x over the communication channel. This causes a huge communication overhead. Therefore, to reduce the communication burden, in this paper the state and input information are transmitted and actuated aperiodically. The aperiodic time-sequence of information transmission is denoted by t_k where $k \in \{1,2,\cdots,N\}$. Due to the aperiodic nature of information transmission, the continuous control input $u(t)$ is converted to $u(t_k)$. Applying $u(t_k)$, the closed-loop system (1) and the HJB equation (6) reduce to

$$\dot{x} = A(x)x + B(x)u(t_k), \tag{10}$$

$$\begin{aligned}-\frac{\partial J^*}{\partial t} =& \frac{1}{2}\big\{x(t)^T Q x(t) + u^*(t_k)^T R u^*(t_k)\big\} + \\ & \left(\frac{\partial J^*}{\partial x}\right)^T \Big\{A(x)x(t) + B(x)u^*(t_k)\Big\}.\end{aligned} \tag{11}$$

From [3] and [6], system (10) can be modeled as a continuous time perturbed system by introducing an error variable $e(t)$. This error variable is used to derive the triggering condition and is defined as

$$e(t) = x(t_k) - x(t),\ \forall t \in (t_k, t_{k+1}). \tag{12}$$

At the event-triggering instant t_k, the numerical value of the measurement error $e(t)$ is zero. Using $e(t)$, the event-triggered control input $u(t_k)$ reduces to

$$\begin{aligned}u(t_k) = K(x,e) = & - B(x(t_k))^T \frac{\partial J^*(x(t),t)}{\partial x(t)}\bigg|_{t=t_k}, \\ & \forall t \in (t_k, t_{k+1}).\end{aligned} \tag{13}$$

The following assumptions should hold for the closed-loop system (10):

Assumption 1: The event-triggered closed-loop system (10) is Lipschitz continuous with respect to state $x(t)$ and measurement error $e(t)$, that means

$$\|A(x)x(t) + B(x)K(x,e)\| \le L_1\|x(t)\| + L_2\|e(t)\|, \tag{14}$$

where L_1 and L_2 are positive constants.

To generate (13), it is essential to solve or approximate the ET-HJB (11). Several researchers have used neural networks (NN) as a universal function approximator to estimate the optimal value function J^* [16]-[17]. To approximate J^*, they update the NN weight vector aperiodically to reduce the computation burden. Apart from NN-based approximation techniques, in [18] Heydari et al. have approximated the equation (6) as an SDRE for a class of nonlinear systems with a stabilizing control law (9). But with an event-triggered control input $u(t_k)$, the conversion of the ET-HJB (11) to an SDRE is a non-trivial problem. To achieve a modified SDRE from (11), the conversion procedure and the numerical tools for solving the proposed SDRE are discussed in the next section. For the stability analysis of the closed-loop event-triggered system (10), the ISS property is adopted. The definition of the ISS property is introduced for a general nonlinear system in [20], which is briefly discussed below.

Definition 1: A system

$$\dot{x}(t) = f(x(t), u(t)), \tag{15}$$

is globally ISS if it satisfies

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma\Big(\sup_{\tau\in[0,\infty)}\{\|u_\tau\|\}\Big), \tag{16}$$

with each input $u(t)$ and each initial state $x(0)$. The functions β and γ are $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions respectively.

Definition 2: Suppose the origin is an equilibrium point of a continuous-time system $\dot{x} = f(x(t), u(t))$, i.e., $f(0,0) = 0$, $\forall\ t > 0$. A positive continuous function $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$ is an ISS Lyapunov function for that system if there exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a class $\mathcal{K}$ function γ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ by satisfying the following conditions.

$$\alpha_1(\|x(t)\|) \le V(x(t)) \le \alpha_2(\|x(t)\|), \tag{17}$$

$$\dot{V}(t) \le -\alpha_3(\|x(t)\|) + \gamma(\|u(t)\|). \tag{18}$$

Problem Statement: Convert the ET-HJB equation (11) into an SDRE and design a state feedback event-triggered control

$$u(t_k) = K\{B(x(t_k)), P(x(t_k)), x(t_k)\}, \tag{19}$$

where the vector $x(t_k)$ and the matrix $P(t_k)$ are the aperiodic system state and the positive definite solution of the proposed Riccati equation. Derive the control law (19) in order to ensure the ISS of the closed-loop system (10) for a given event-triggering rule.

III. Main Results

The primary theoretical contributions of this paper are divided into three parts, and they are discussed in the following two subsections. First, conversion of the ET-HJB to an SDRE is discussed, and then using the solution of the SDRE, the stability of the closed-loop system is ensured. Along with this stability result, a state-dependent event-triggering rule is also defined. Finally, a numerical procedure to solve the proposed SDRE is reported. The following standard assumption [20], [21] is used to convert (11) into an SDRE.

Assumption 2: [16] For a positive constant L and measurement error $e(t)$, the event-triggered control $u(t_k)$ and continuous control $u(t)$ hold the following inequality

$$\|u(t) - u(t_k)\| \le L\|e(t)\|,\ \forall t \in (t_k, t_{k+1}). \tag{20}$$

A. Conversion of the ET-HJB into an SDRE

The conversion process of the ET-HJB into an SDRE is stated next.

Theorem 1: Suppose there exist positive scalars L and $\sigma \in (0,1)$ such that Assumption 2 holds. For an event-triggering rule

$$t_0 = 0,\ t_{k+1} = inf\{t \in \mathbb{R} | t > t_k \wedge (\mu\|x\|^2 - \|e\|^2 \leq 0)\}, \tag{21}$$

the ET-HJB (11) reduces to the following SDRE

$$\begin{aligned}-\dot{P}(x,t) \leq P(x,t)A(x) + A^T(x)P(x,t) - \\ P(x,t)B(x)B(x)P(x,t) + (\sigma+1)Q - \mho,\end{aligned} \tag{22}$$

where scaling matrix $R = I$ and the boundary condition $P(x,t_f) = F$ are selected. The selection $R = I$ simplifies the expression however, one can select any value for R. The parameter μ and matrix $\mho$ are defined by

$$\mu = \frac{\sigma \lambda_{\min}(Q)}{L^2}, \tag{23}$$

$$\mho = \left\{ \sum_{i=1}^{n} P_{x_i} z_i + \frac{1}{4}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} P_{x_i} x (B(x)B(x)^T)_{ij} x^T P_{x_j} \right) + (B(x)B(x)^T) \begin{bmatrix} x^T \frac{\partial P(x,t)}{\partial x_1} \\ \vdots \\ x^T \frac{\partial P(x,t)}{\partial x_n} \end{bmatrix} \right\}, \tag{24}$$

where the scalar $z_i = b_i u(t_k)$ and b_i is the i^{th} element of input matrix $B(x)$. The notation $(\ \cdot\ )_{ij}$ represents the ij^{th} element of matrix $(\ \cdot\ )$ The partial derivative of a positive matrix $P(x,t)$ with respect to the individual state-element $x_{i\in\{1\leq i\leq n\}}$ is denoted by P_{x_i}.

Proof: Using $R = I$, the ET-HJB (11) reduces to

$$-\frac{\partial J^*}{\partial t} = \frac{1}{2}(x^T Q x + u(t_k)^T u(t_k)) + \left(\frac{\partial J^*}{\partial x}\right)^T \left\{ A(x)x(t) + B(x)u(t_k) \right\}. \tag{25}$$

which has an optimal cost-to-go

$$J^*(x,t) = x^T P(x,t) x. \tag{26}$$

The partial derivative of $J^*(x,t)$ with respect to t and x are simplified as

$$\frac{\partial J^*}{\partial t} = \frac{1}{2} x^T P_t x, \tag{27}$$

$$\frac{\partial J^*}{\partial x} = Px + \Gamma, \tag{28}$$

where $P_t = \frac{\partial P}{\partial t}$ and

$$\Gamma = \frac{1}{2}\begin{bmatrix} x^T \frac{\partial P(x,t)}{\partial x_1} x \\ \vdots \\ x^T \frac{\partial P(x,t)}{\partial x_n} x \end{bmatrix}. \tag{29}$$

Using (27) and (28), the ET-HJB (25) reduces to

$$\begin{aligned}-\frac{1}{2}x^T P_t x = \frac{1}{2}\{x(t)^T Q x(t) + u(t_k)^T u(t_k)\} \\ +(Px+\Gamma)^T \{A(x)x + B(x)u(t_k)\}.\end{aligned} \tag{30}$$

Further, (30) is simplified as

$$\begin{aligned}-\frac{1}{2}x^T P_t x - \Gamma^T \dot{x} = \frac{1}{2}\{x^T Q x + u(t_k)^T u(t_k)\} + \\ x^T P(x,t)A(x)x + x^T P(x,t)B(x)u(t_k).\end{aligned} \tag{31}$$

Equation (31) can be written as

$$\begin{aligned}-\frac{1}{2}x^T \dot{P}(x,t) x = \frac{1}{2}\{x^T Q x + u(t_k)^T u(t_k)\} + \\ x^T P(x,t)A(x)x + x^T P(x,t)B(x)u(t_k),\end{aligned} \tag{32}$$

where the following auxiliary equations are used

$$\Gamma^T \dot{x} = \frac{1}{2} x^T (\sum_{i=1}^{n} P_{x_i}\dot{x}_i) x,$$

$$-\frac{1}{2}x^T P_t x - \frac{1}{2}x^T(\sum_{i=1}^{n} P_{x_i}\dot{x}_i) x = -\frac{1}{2}x^T \dot{P} x.$$

From (5) and (28), the optimal input $u^*(t)$ can be written as

$$u^{*T}(t) = -(Px+\Gamma)^T B(x). \tag{33}$$

For simplicity of notation, the optimal cost-functional J^*, the matrices $P(x,t)$, $A(x)$, $B(x)$ and input $u^*(t)$ are now denoted by J, P, A, B and $u(t)$ respectively. Applying (33), the term $x^T PB(x)u(t_k)$ is simplified as

$$\begin{aligned} x^T P B u(t_k) &= (Px+\Gamma)^T B u(t_k) - \Gamma^T B u(t_k) \\ &= -u^T(t) u(t_k) - \Gamma^T B u(t_k). \end{aligned} \tag{34}$$

Using (34), equation (32) reduces to the following equality

$$\begin{aligned}-\frac{1}{2}x^T \dot{P} x = &\frac{1}{2}\{x^T Q x + u(t_k)^T u(t_k)\} + x^T P A x \\ &+ \frac{1}{2}(u(t) - u(t_k))^T (u(t) - u(t_k)) \\ &- \frac{1}{2}\{u(t)^T u(t) + u(t_k)^T u(t_k)\} - \Gamma^T B u(t_k).\end{aligned} \tag{35}$$

The event-triggering rule (21) is used to define an upper bound of (20). Now using (20) and (21), the equality (35) reduces to following inequality

$$-\frac{1}{2}x^T \dot{P} x \leq \frac{(\sigma+1)}{2} x^T Q x + x^T P A x - \frac{1}{2}u^T u - \Gamma^T B u(t_k). \tag{36}$$

Equation (33) is used to simplify the inequality (36):

$$\begin{aligned}-x^T \dot{P} x \leq (\sigma+1)x^T Q x + x^T P A x + x^T A^T P x - \\ x^T P B B^T P x - [2x^T P B B^T \Gamma + \Gamma^T B B^T \Gamma + 2\Gamma^T B u(t_k)].\end{aligned} \tag{37}$$

To construct a Riccati-like equation, it is essential to express the terms $2\Gamma^T B u(t_k)$, $\Gamma^T B(x)B(x)^T \Gamma$ and $2x^T P B(x) B(x)^T \Gamma$ in quadratic form. To that aim, the following simplifications are adopted:

- The event-triggered input $u(t_k)$ is constant within the inter-event time. The input matrix $B(x) = \begin{bmatrix} b_1(x) & b_2(x) & \cdots & b_n(x) \end{bmatrix}^T$ depends on the system's sate x. Therefore, the term $B(x)u(t_k)$ can be represented by another different state dependent variable $Z(x)$ and it is defined as follows

$$Z(x) = \begin{bmatrix} z_1(x) & z_2(x) & \cdots & z_n(x) \end{bmatrix}^T, \quad (38)$$

where $z_i(x) = b_i(x)u(t_k)$ for $i = 1, 2 \ldots n$. Using (38) and (29), the following equalities are obtained

$$2\Gamma^T Bu(t_k) = x^T(\sum_{i=1}^{n} P_{x_i} z_i)x, \quad (39)$$

$$\Gamma^T BB^T \Gamma = \frac{1}{4} x^T(\sum_{i=1}^{n}\sum_{j=1}^{n} P_{x_i} x(BB^T)_{ij} x^T P_{x_j})x. \quad (40)$$

- Similarly, the term $2x^T PBB^T\Gamma$ is reduced to

$$2x^T PBB^T\Gamma = x^T PBB^T \begin{bmatrix} x^T \frac{\partial P(x,t)}{\partial x_1} \\ \cdots \\ x^T \frac{\partial P(x,t)}{\partial x_n} \end{bmatrix} x. \quad (41)$$

Using (39), (40) and (41), the equation (37) is simplified as

$$\begin{aligned} -x^T\dot{P}x \leq & x^T PAx + x^T A^T Px - x^T PBB^T Px + \\ & (\sigma+1)x^T Qx - x^T \mho x \end{aligned} \quad (42)$$

where the matrix $\mho$ is defined in (24). Now rewriting the inequality (42), (22) is obtained for a final condition $P(x, t_f) = F$. That completes the proof. ∎

Remark 1: Equation (22) is converted into an approximated Riccati equation by neglecting the term $\mho$ from (22). The expression of the approximated Riccati equation is

$$\begin{aligned} -\dot{P}(x,t) = & (\sigma+1)Q + P(x,t)A(x) + A^T(x)P(x,t) - \\ & P(x,t)B(x)B(x)^T P(x,t) \end{aligned} \quad (43)$$

with a boundary condition $P(x, t_f) = F$. The approximation helps to solve the Riccati equation numerically by avoiding the partial differential terms exist in $\mho$. The constant matrix $(\sigma + 1)Q > 0$ in (43) is changed from (7) due to the aperiodic update of control inputs. The solution P of (43) is used to compute the event-triggered control input (19). The approximate control input will not be optimal, but the ISS property holds for (10).

The following theorem ensures the ISS of (10) for the event-triggering law (21).

Theorem 2: Suppose there exists a positive definite solution $P(x, t)$ of (43) which is aperiodically computed for an event-triggering rule (21). The solution $P(x, t)\big|_{t=t_k}$ and aperiodic state information $x(t_k)$ generate the event-triggered control input

$$u(t_k) = -B(x(t_k))^T P(x(t_k), t_k)x(t_k), \quad (44)$$

which is actuated at the system end based on the event-triggering rule (21). The control input (44) ensures the ISS of the closed-loop system (10).

Proof: Defining an ISS Lyapunov function $V(x) = x^T P(x, t)x$, the time derivative of $V(x)$ along the solution of (10) is

$$\begin{aligned} \dot{V}(x) = & -(\sigma+1)x^T Qx + (u(t) - u(t_k))^T(u(t) - u(t_k)) - \\ & u(t_k)^T u(t_k). \end{aligned} \quad (45)$$

After further simplification, the upper bound of (45) is

$$\dot{V}(x) \leq -(\sigma+1)x^T Qx + (u(t) - u(t_k))^T(u(t) - u(t_k)). \quad (46)$$

Applying Assumption 2 to (46), the following equation is obtained

$$\dot{V}(x) \leq -\sigma x^T Qx + L^2\|e\|^2. \quad (47)$$

Using Definition 2, (47) ensures the ISS of (10) for an input (44) with the event-triggering rule (21). ∎

Remark 2: From (40) and (24), it is possible to show that there exist two constants c_1 and c_2 such that the Γ and $\mho$ are bounded by the following equations

$$\Gamma \leq c_1 \|x\|^2, \quad (48)$$

$$\mho \leq c_2 \|x\|^2. \quad (49)$$

Therefore as $x(t) \to 0$ for $t \to \infty$, the terms Γ and $\mho$ also approach towards zero. This also ensures the convergence of control input (44) as $t \to \infty$ [26].

Remark 3: In the SDC form of (10), the state and input matrices depend on the state information. However, continuous state information is not available in the controller due to communication constraints. To resolve this problem, we have adopted the frozen-time concept borrowed from [22] to solve (43). It assumes that the state and input matrices remain constant in-between two consecutive events. This also helps to solve the Riccati equation in a frozen-time manner.

Remark 4: In event-triggered control it is essential to prove that the inter-event time $\tau = (t_{k+1} - t_k)$ is always positive i.e. $\tau > 0$. This constraint is imposed to avoid the Zeno behavior [1] in system dynamics. Now for the system (10) whose initial condition $x(0)$ remains in a compact set $S \subseteq \mathbb{R}^n$ (i.e. $x(0) \in S$), there exists a lower bound $\tau \in \mathbb{R}^+$ for the event-triggering rule (21) which satisfies $t_{k+1} - t_k \geq \tau$, $\forall k \in \mathbb{N}$. This can be proved in the similar way as Theorem III.1 in [3].

B. Numerical solution of SDRE

As per Remark 3, the derived SDRE (43) can be considered as the following DRE within the two consecutive events [22]

$$\begin{aligned} -\dot{P}(t, x(t_k)) = & P(t)A(x(t_k)) + A^T(x(t_k))P(x(t_k), t) - \\ & P(x(t_k), t)B(x(t_k))B(x(t_k))^T P(x(t_k), t) + (\sigma+1)Q. \end{aligned} \quad (50)$$

The solution procedure of (50) is discussed next [23]:

[1] Infinite number of transmission and computation in a finite time [25].

- Compute the steady-state value (P_{ss}) of (50) , by solving the following algebraic Riccati equation:

$$P_{ss}A + A^T P_{ss} - P_{ss}BB^T P_{ss} + (\alpha+1)Q = 0. \quad (51)$$

- Subtracting (50) from (51), the following equation is obtained

$$\begin{aligned} -\dot{P}(t) =& (P - P_{ss})A + A^T(P - P_{ss}) \\ & - P(t)SP(t) + P_{ss}^T S P_{ss}. \end{aligned} \quad (52)$$

where matrix $S = B(x(t_k))B(x(t_k))^T$.
- Using the change of variable $P_0(t) = (P(t) - P_{ss})^{-1}$, equation (52) is recast as a differential Lyapunov equation [24]:

$$\dot{P}_0(t) = A_{cl}P_0(t) + P_0(t)A_{cl}^T - S, \quad (53)$$

with the final condition $P_0(t_f) = (F - P_{ss})^{-1}$ and $A_{cl} = A(x(t_k)) - SP_{ss}$.
- Compute the solution of the algebraic Lyapunov equation

$$A_{cl}D_0 + D_0A_{cl}^T - S = 0. \quad (54)$$

- The solution of (53) is

$$P_0(t) = e^{A_{cl}(t-t_f)}(P_0(t_f) - D_0)e^{A_{cl}^T(t-t_f)} + A_0. \quad (55)$$

- The solution of original SDRE (50) is

$$P(t) = P_{ss} + P_0(t)^{-1}. \quad (56)$$

The expression (56) is used to compute (44).

C. Comparison with existing work

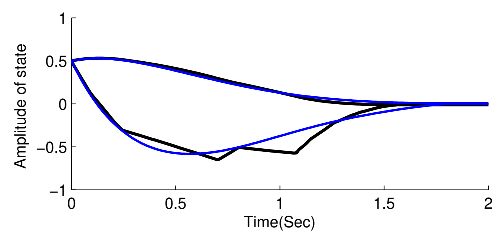
Here, we compare the main contributions of this paper with the existing research work [18]. In [18], the HJB equation has been approximated as an SDRE. To approximate the HJB equation into an SDRE, the primary assumption in the work is that the state and input information are continuously available to the controller and system end, respectively. In general, this assumption does not hold for event-triggered control techniques due to the asynchronous availability of state and input information. In [16], K. G. Vamvoudakis has named the equation (11) as ET-HJB equation for the presence of aperiodic control input $u(t_k)$. This paper proposes a procedure to convert the ET-HJB equation into an SDRE which helps solve the nonlinear optimal control problem for a class of systems with limited feedback information. The presence of aperiodic input $u(t_k)$ in (11), complicates the conversion processes of the ET-HJB equation into an SDRE. The detailed steps of conversion processes are described in this paper [refer Theorem 1 and its proof]. Due to the limited availability of state and control input, we obtain a state-dependent differential Riccati-like inequality (22). The inequality (22) is converted to a Riccati equation (43), using an equality relation. The positive definite solution P of (43) is evaluated at every event-triggering instant (t_k), to compute the aperiodic control input $u(t_k)(= -B(x(t_k))^T P(x(t_k), t_k)x(t_k))$. To handle the aperiodic actuation of control input $u(t_k)$, the constant matrix Q of (43) is scaled by $(1+\sigma)$ times compared to reported work [18]. In [18], the solution P of Riccati equation [equation (15) in [18]] has been computed continuously, but in this paper a frozen-time approach is adopted to solve (43) aperiodically.

IV. Results

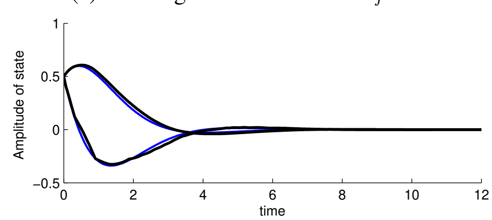
In this section, a benchmark example, the regulation problem of *Van der Pol's Oscillator* system, is considered to illustrate the efficacy of the proposed control algorithm. Validation of the control problem is solved numerically. The *Van der Pol's Oscillator* model is described as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & x_2\\ -1 & 1-x_1^2\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u. \quad (57)$$

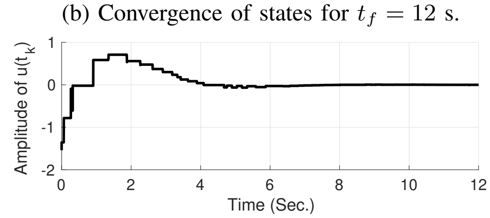
For simulation, matrices $Q = I$, $R = I$, and $F = 10I$ are selected. To realize the event-triggering law (44), the scalar L and design parameter σ are considered as 0.82 and 0.1 respectively. The simulation is carried out in Matlab with a final time $t_f =$ 2 and 12 sec. The initial states are selected as $\begin{bmatrix}0.5 & 0.5\end{bmatrix}^T$. The Figures 2a, and 2b show the time evolution of states for different final time t_f. The black and blue lines are used to represent the state trajectories of the system using the event-triggered and continuous control input respectively. From these Figures, it is observed that the proposed event-based control input helps to bring the system states near to the equilibrium point in a finite-time. The aperiodic variation of control input $u(t_k)$ is shown in Figure 2c. The numerical results show the ISS of closed loop system (10) and ensure the finite-time convergence of state trajectories for an aperiodic update of control input



(a) Convergence of states for $t_f = 2$ s.



(b) Convergence of states for $t_f = 12$ s.



(c) Variation of control input $u(t_k)$ for $t_f = 12$ s.

Fig. 2: Results of continuous and event-triggered control.

$u(t_k)$. From Table I it is observed the total number of input computation for continuous approach is comparatively higher than the event-triggered one. The minimum and maximum bound of inter-event time [τ_{min} and τ_{max} respectively] of proposed event-triggering rule are shown in Table I. The notation u_{total} represents the number of input computation in total runtime.

Control mechanism		Performance for $t_f = 12$ sec.		
		$\tau_{\text{max}}(s)$	$\tau_{\text{min}}(s)$	u_{total}
Results	Continuous control	0.01	0.01	1200
	Event-triggered control	0.7149	0.01	144

TABLE I: Comparative results of event-triggered and continuous control approach

V. Conclusion

This paper proposes a novel finite-time event-triggered control law for a class of nonlinear systems. An event-triggered control law is derived based on the aperiodic state information to reduce the bandwidth requirement. The control law is designed without solving the ET-HJB equation explicitly. It is possible by converting the HJB equation into a modified SDRE which is an ordinary differential equation. The solution of the SDRE equation is obtained by solving DLE. The solution of SDRE is computed in an aperiodic manner to reduce the computation burden. The numerical results help to validate the proposed algorithm. This framework has promising future direction in different control and estimation problems like in adaptive control [16] , robust control [28] and multi-sensor state estimation [32] under limited feedback information. In many practical situations, e.g., guidance path planning, the control law should achieve the desired goal in a certain predefined time with aperiodic feedback. The proposed SDRE based approach has a potential application for such problems [26].

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