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RESEARCH

# Extremely short 3D Bessel pulses in an optically anisotropic photonic crystal made from CNTs with account for nonlinear absorption and pumping

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**Abstract**
We present a theoretical and numerical study of the dynamics of diffractionless three-dimensional (3D) extremely short optical pulses with a Bessel cross-section in an optically anisotropic medium with a spatially modulated refractive index (known as photonic crystal) based on carbon nanotubes of the zig-zag type. The nonlinear absorption is taken into account based on experimental data, as well as the effects associated with the external pump field. Through numerical simulations, it was found that in such an optically anisotropic photonic crystal, Bessel pulses propagate stably over times of several dispersion lengths, while dissipative solitons arise during the evolution of pulses. The influence of the parameters of the anisotropic photonic crystal—e.g., the depth and period of modulation of the refractive index and anisotropy parameters—on the evolution of pulses, in particular, their shape and group velocity, is established.

## 1 Introduction

One of the pressing problems of modern nonlinear optics and photonics is the study of the interaction of optical radiation with various media exhibiting a range of nonlinear properties. Among of these promising media there are photonic crystals, which are essentially superlattices in which there is an additional modulation of the refractive index with a characteristic scale of periodicity of the order of the wavelength of a pulse passing through it [1]. The presence of photonic band gaps in the spectrum of electromagnetic excitations of a photonic crystal offers an interesting opportunity to manipulate and control the speed of optical radiation propagating through it. This unique combination of localization of electromagnetic radiation and control of its parameters—specifically, the shape of the envelope as well as the group velocity of the wave packet—distinguishes photonic crystal materials from any of the previously studied optical media [2]. It is worth stressing that the variability in the refractive index of a photonic crystal provides an ideal nonlinear medium for the study of the propagation of electromagnetic solitons, extremely short optical pulses and light bullets [3].

By extremely short optical pulses (ESOP), we mean femtosecond pulses which contain a small number of periods of electric field oscillations. The energy of such pulses remains localized in a limited region of space [4, 5]. It should be noted that ESOP have a high directivity of their radiation and shape stability; they are strongly resistant to disturbance in a range of parameters [6–8].

To compensate for the dispersive spreading of ESOP, it is usually necessary for the medium to have nonlinear properties. However, non-diffracting and non-dispersive solution, so-called Focus modes, are known since 2002 [9, 10]. Note, those modes are dispersion resistant. In this papers, we consider carbon nanotubes (CNTs) with their parabolic electron dispersion law as a photonic crystal [11]. The nonparabolicity of the electron dispersion law that induces the

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nonlinearity of the response of CNTs to the action of electromagnetic pulses [12, 13]. However, in the presence of CNTs, additional dissipative effects arise and therefore some form of compensation is required, i.e., energy "pumping".

To circumvent the undesirable diffraction effects, we consider ESOP having a Bessel cross-section. Such pulses received a particular attention among both theorists and experimenters owing to their unique property of non-diffractionality, i.e., ESOP spread without refraction or scattering [14]. Bessel pulses can be formed in nonlinear media, the refractive index of which changes weakly in a periodic manner depending on the wavelength [15]. In practice, Bessel pulses have been observed in nonlinear crystals, as shown in Refs. [16, 17]. There are well-known cases when Bessel pulses are formed in nonlinear media, which are described, for example, in Refs. [18–20].

The present study has certain prerequisites previously detailed in Refs. [21–23]. Moreover, the propagation of ESOP in a medium made of a photonic crystal based on CNTs has been previously reported [24], while the influence of the external pumping field has been accounted for in Ref. [25, 26]. Lastly, the dynamics of diffractionless Bessel pulses in a medium with a variable refractive index is described in Ref. [27, 28]. Another interesting yet challenging problem is taking into account the optical anisotropy of the medium so as to indirectly be able to control the pulse propagation in it. Indeed, the anisotropy of the medium can lead to various interesting effects, such as the Zakharov–Benny resonance for instance [29].

Nonetheless, it is of high relevance to consider dissipative effects, which are inevitable with any real devices. In particular, the problem of considering the compounded effects all parameters that stabilize or annihilate extremely short pulses at large times is nontrivial. Here, we partially address this challenging problem.

## 2 Governing equations

The geometry of the problem under consideration assumes that the pulse propagates along the direction of the $Oz$-axis, which is the direction over which the refractive index of the photonic crystal is spatially modulated. The applied electric field is directed at an angle with the CNT axis (see Fig. 1 [30]).

It should be emphasized that in the present study we neglect the electric field of the substrate. In addition, we use the continuous medium approximation; it is assumed that the current is uniformly distributed over the volume of the photonic crystal. This approximation is valid as far as the geometric dimensions of nanotubes and the distance between them are at least an order of magnitude smaller than the size of the region of localization of the electric field of the pulse.

With the gauge $\mathbf{E} = -c^{-1}\partial\mathbf{A}/\partial t$, Maxwell's equations for the vector potential component of the electric field of the pulse take the form

$$\frac{\partial^2\mathbf{A}}{\partial x^2} + \frac{\partial^2\mathbf{A}}{\partial y^2} + \frac{\partial^2\mathbf{A}}{\partial z^2} - \frac{n^2(z)}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} + \Gamma\frac{\partial\mathbf{A}}{\partial t} - F_1\left(\frac{\partial\mathbf{A}}{\partial t}\right)\left|\frac{\partial\mathbf{A}}{\partial t}\right|^2 - \frac{F_2\frac{\partial\mathbf{A}}{\partial t}}{1+\Delta\left(\frac{\partial\mathbf{A}}{\partial t}\right)^2} = 0, \tag{1}$$

where $\mathbf{A} = \{A_x(x,y,z,t), A_y(x,y,z,t), 0\}$ is the vector potential, $n(z) = 1 + \mu\cos(2\pi z/\chi)$ is the spatially modulated refractive index of the medium, which specifies an optically anisotropic photonic crystal ($\mu$ is the modulation depth of the refractive index and $\chi$ its modulation period), $c$ is the

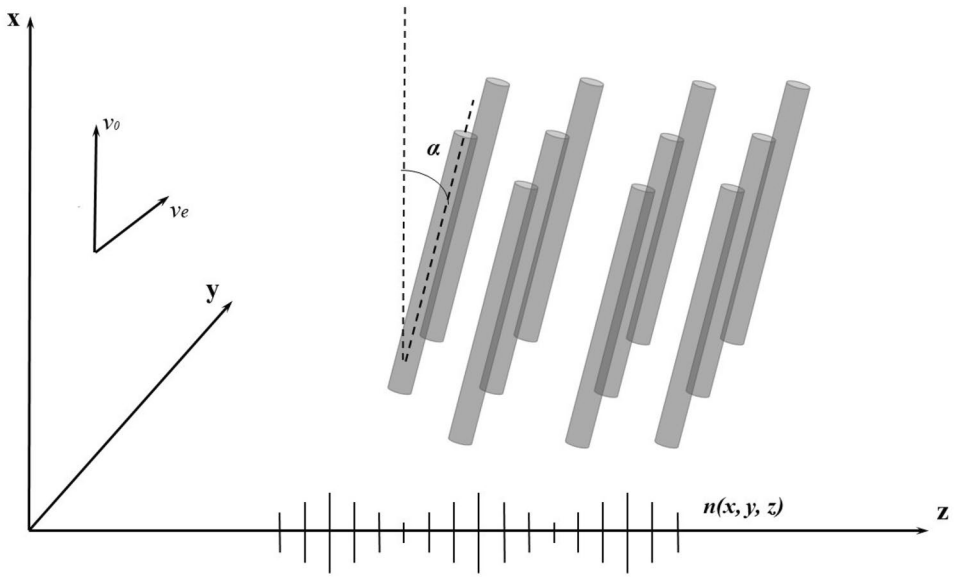


**Fig. 1** Geometry of the system. $v_0$ is the speed of an ordinary beam, $v_e$ is the speed of an extraordinary beam, $\alpha$ is the angle between the CNT axis and the vector potential of the electric field of the pulse. The dashes represent the spatial modulation of the refractive index along the $z$ direction

speed of light in vacuum, Γ is the index of pumping/amplification of the electric field [31] (without linear absorption), $\mathbf{j} = \{j_x(x,y,z,t), j_y(x,y,z,t), 0\}$ is the electric current density, $F_1$ and $F_2$ are the nonlinear absorption coefficients usually determined experimentally [32], and Δ is the nonlinear absorption saturation parameter. The last term in equation (1) is taken from the experiment [32] and describes the nonlinear damping for a system of carbon nanotubes.

The current density, which originates from the interaction of the pulse field with electrons in the conduction band of CNTs, has the form [33–36]

$$\begin{aligned} j &= 2e\sum_{s=1}^{m}\int_{\mathrm{BZ}} v_s(p) f(p,s)\,\mathrm{d}p,\\ v_s(p) &= \frac{\partial \varepsilon_s(p)}{\partial p},\\ \varepsilon_s(p) &= \pm\gamma_0\left\{1+4\cos(ap)\cos(\pi s/m)+\cos^2(\pi s/m)\right\}^{1/2}, \end{aligned} \tag{2}$$

where $v_s(p)$ is the group velocity of electrons, $e$ is the electron charge, $f(p, s)$ is the distribution function, which at the initial moment of time coincides with the Fermi distribution function, BZ stands for the first Brillouin zone, $\varepsilon_s(p)$ is the dispersion law of $\pi$-electrons in the semiconductor nanotubes [12], $\gamma_0$ is the overlap integral ($\approx 2.7$ eV), $a$ is the lattice constant of CNTs, $p$ is the quasimomentum of the electron, and $s = 1, \ldots m$ is the number of hexagons along the nanotube perimeter.

Taking into account the dispersion law, the group velocity can be expanded as a Fourier series:

$$\begin{aligned} v_z(s,x) &= \sum_m a_{ms}\sin(mx),\\ a_{ms} &= \frac{1}{\pi}\int_{-\pi}^{\pi} v_z(s,x)\sin(mx)\,\mathrm{d}x, \end{aligned} \tag{3}$$

where the Fourier coefficients $a_{ms}$ decrease with increasing $m$. Here, we can restrict ourselves to the first 15 nonvanishing terms [36] and obtain the generalized sine-Gordon equation, widely used in applications, but not integrable by the inverse scattering problem method.

To compensate for the dissipative effects and the diffraction spreading of the pulse outside the amplifying region, external energy is pumped into the system and this is controlled by means of the coefficient Γ. The action of the external field is assumed to be concentrated in a region near the axis; the shape of the region where the pumping is primarily effective is chosen to be that of a super-Gaussian form [37].

Taking into account the expressions of the current density (2) and the group velocity (3), the equation for the components of the vector potential of the electric field in cylindrical coordinates can be cast as [36, 38]

$$\begin{aligned} &\frac{\partial^2 A_{x,y}}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_{x,y}}{\partial r}\right)-\frac{n^2(z)}{c^2}\frac{\partial^2 A_{x,z}}{\partial t^2}\\ &+\Gamma\frac{\partial A_{x,y}}{\partial t}-F_1\left(\frac{\partial A_{x,y}}{\partial t}\right)^3-\frac{F_2\frac{\partial A_{x,y}}{\partial t}}{1+\Delta\left(\frac{\partial A_{x,y}}{\partial t}\right)^2}\\ &+\frac{4en_0\gamma_0 a}{c}\sin\alpha\sum_{q=1} b_q\frac{aeq}{c}\phi(t)\cos\left[\frac{aeq}{c}\left(A_x\cos\alpha+A_y\sin\alpha\right)\right]=0, \end{aligned} \tag{4}$$

where

$$r=\sqrt{x^2+y^2},\qquad b_q = \sum_{s=1} a_{sq}\int_{BZ}\cos(pq)\frac{\exp\left(-\varepsilon_s(p)/k_BT\right)}{1+\exp\left(-\varepsilon_s(p)/k_BT\right)}\mathrm{d}p.$$

Here, $n_0$ is the electron concentration; $a_{sq}$ are the Fourier coefficients of the dispersion law of electrons [39], $k_B$ is the Boltzmann constant, and $T$ the temperature.

Note that the solution of the Boltzmann equation with collision terms in the relaxation time approximation leads to an exponential decrease in the current as a function of time [40, 41]

$$\phi(t)=\begin{cases} 0, & t<t_0(z)\\ \exp(-t/t_{\mathrm{rel}}), & t\geq t_0(z)\end{cases}, \tag{5}$$

where $t_0(z) \simeq (z - z_0)/v$ is the moment in time at which the intensity of the pulse at its leading edge, measured at a point with the coordinate $z$, is $e$ times less than the peak intensity of the pulse; $z_0$ is the coordinate of the "center of mass" of the pulse at the initial moment of time; $v \simeq c/\sqrt{k_0}$ is the approximation of the pulse velocity in order of magnitude; $k_0$ is the average relative permittivity of the optically anisotropic photonic crystal, and $t_{\mathrm{rel}}$ is the relaxation time of the electron subsystem of the CNT. Note that $\phi(t)$ is a correction factor to the expression for the current density in the collisionless approximation, which is written empirically based on the generalization and analysis of the results of numerical simulation and is an approximation for the current density in the relaxation time approximation. In other worlds, a current pulse induced by an extremely short optical pulse decays exponentially when relaxation processes are taken into account.

Since the field of an ESOP propagating in a photonic crystal made of CNTs is inhomogeneous, current inhomogeneity may also occur, which in turn can lead to a charge accumulation in some regions. However, calculations in Ref. [39] have shown that the effect of charge accumulation for femtosecond pulses is negligible. As a result, we can assume that the cylindrical symmetry is preserved in

the field distribution and, therefore, the derivative with respect to the angle can also be neglected.

The initial conditions for the vector potential of the electric field of a three-dimensional ESOP with a Bessel cross-section are as follows:

$$\begin{aligned} A_x|_{t=0} &= A_{0x} \exp\left\{-\left(\frac{z}{\gamma_z}\right)^k\right\} J_0 \\ &\quad \times \left(\left|\frac{r-r_0}{\gamma_r}\right|\right) \exp(-\delta|r-r_0|), \\ \frac{dA_x}{dt}|_{t=0} &= A_{0x}\frac{2uz}{\gamma_z^2} \exp\left\{-\left(\frac{z}{\gamma_z}\right)^k\right\} J_0\left(\left|\frac{r-r_0}{\gamma_r}\right|\right) \\ &\quad \exp(-\delta|r-r_0|), \\ A_y|_{t=0} &= 0, \qquad \frac{dA_y}{dt}|_{t=0} = 0, \end{aligned} \tag{6}$$

where $\gamma_z$ and $\gamma_r$ are the parameters determining the pulse width along the $z$- and $r$-axis, $t_0$ is the initial moment of time, $u$ is the initial pulse velocity at the entrance to the medium, $J_0$ stands for the zero-order Bessel function of the first kind, $\delta$ is the cutoff parameter, which is introduced due to the fact that the pulse with the Bessel cross-section is physically unrealizable, and the function must be cutoff at large distances to ensure a finite value of the pulse energy.

## 3 Simulation results

The effective equation for the components of the ESOP vector potential (5) is solved numerically using an explicit finite difference scheme of the "cross" type [42]. In the numerical simulations of the system under study, the following values for the parameters were chosen [36]: $m = 13$, $T = 293$ K, relaxation time in CNT $\approx 10^{-11}$ s, pulse duration $\approx 10^{-14}$ s.

The evolution of a diffractionless three-dimensional ESOP with Bessel cross-section during its propagation in an optically anisotropic photonic crystal made of CNTs, taking into account nonlinear absorption and pumping by an external field, is shown in Figs. 2 and 3. Here, we apply the following typical values for the system parameters: refractive index modulation depth $\mu = 0.25$, refractive index modulation period $\chi = 2.5\,\mu$m, the angle between the CNT axis and the pulse electric field vector $\alpha = \pi/3$, the pulse velocities along the crystallographic axes $v_x/c = 0.9$ and $v_y/c = 0.5$; the exponent in the initial conditions is $k = 2$.

Figures 2 and 3 demonstrate that the pulse retains its energy localized in a limited spatial region, the change in the pulse amplitude does not exceed 10%, while its shape undergoes a relatively insignificant change (Fig. 3). Thus, one can speak of a stable pulse propagation and

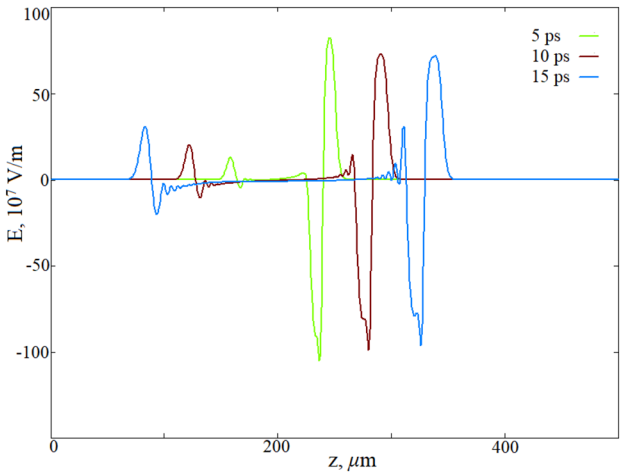


**Fig. 3** Longitudinal sections of the pulses presented in Fig. 2 at $r = 0$

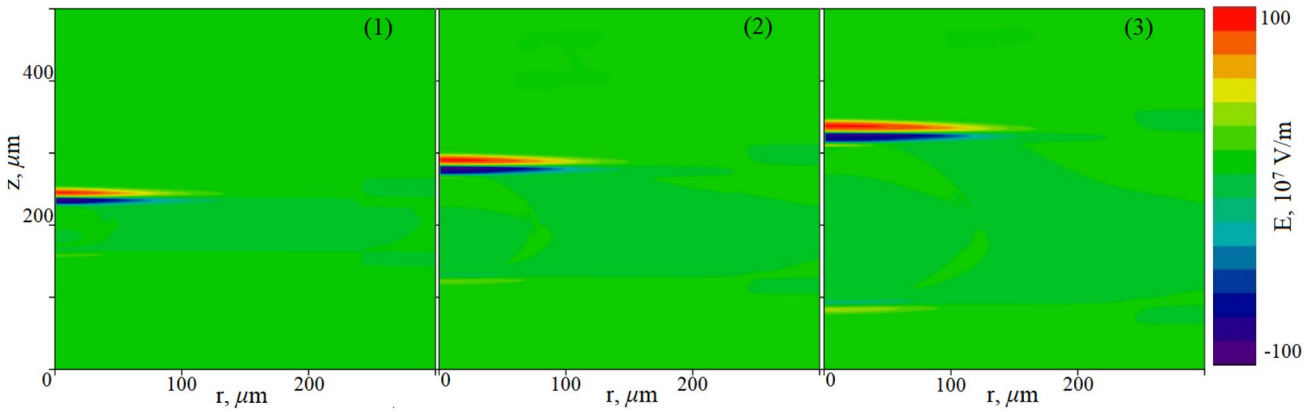


**Fig. 2** Dynamics of a three-dimensional ESOP with a Bessel cross-section in an optically anisotropic photonic crystal made of CNTs at different times: 5 ps, 10 ps, and 15 ps from left to right, respectively

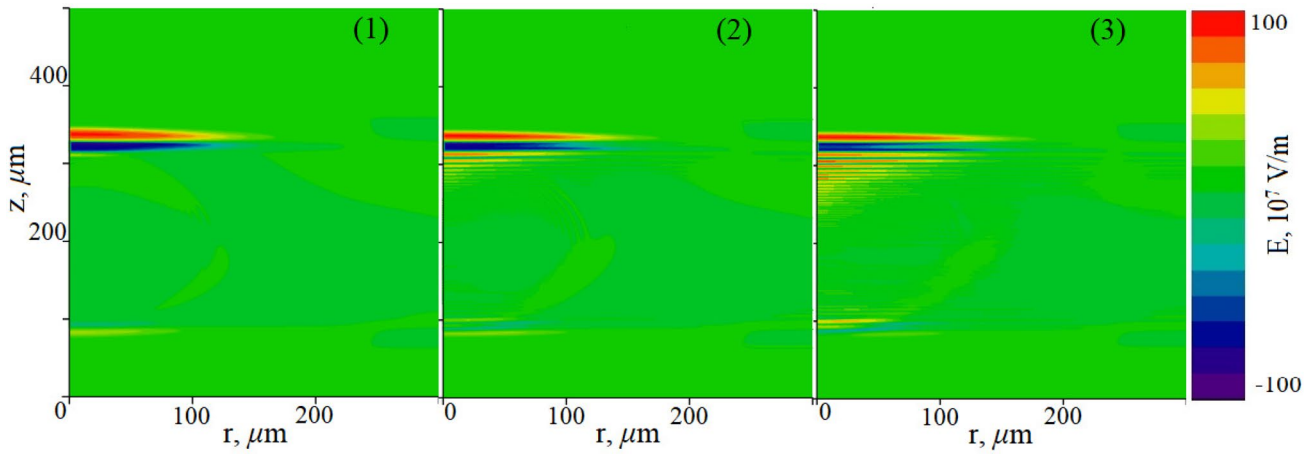


**Fig. 4** Dynamics of a three-dimensional ESOP with a Bessel cross-section in an optically anisotropic photonic crystal made of CNTs at 15 ps for different initial pulse shapes: $k = 2$, $k = 4$, and $k = 6$ from left to right, respectively

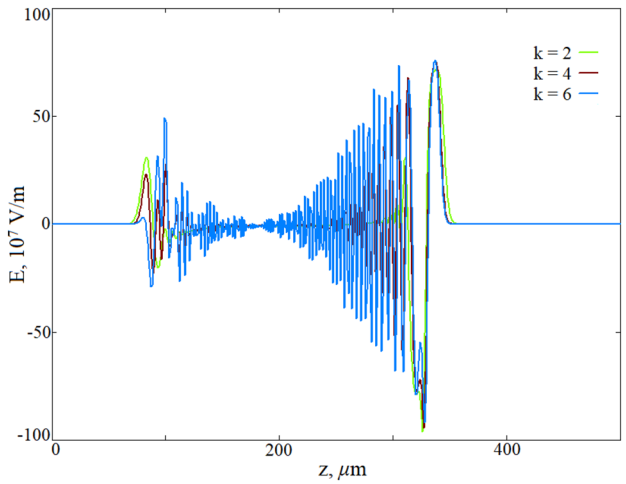


**Fig. 5** Longitudinal sections of the pulses presented in Fig. 4 at $r = 0$

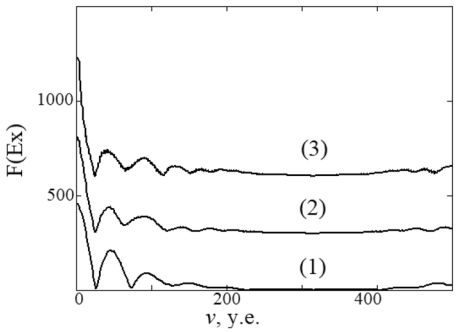


**Fig. 6** Fourier spectra of a three-dimensional diffractionless ESOP with a Bessel cross-section in an optically anisotropic photonic crystal made of CNTs at different times: (1) 5 ps; (2) 10 ps; (3) 15 ps

the presence of a balance between the competing factors that, on the one hand stabilize the pulse—through pumping by an external field and nonlinearity of the photonic crystal—and on the other hand, those that annihilate the pulse—nonlinear absorption, dispersion and diffraction spreading. Dispersion spreading along the sample axis is compensated by the nonlinearity of the medium of CNTS owing to the non-parabolicity of the dispersion law for electrons in the conduction band.

Subsequently, we simulate a pulse at a time instant of 15 ps with a different initial shape. Specifically, the value of the exponent $k$ is changed in the initial conditions. The results are presented in Figs. 4 and 5. The parameters of anisotropy and refractive index are identical to those in the previous case.

Figures 4 and 5 show that a change in the exponent in the initial conditions does not cause changes in the shape and amplitude at the leading edge of the pulse, which in turn indicates the presence of so-called dissipative solitons. Requirements for the properties of the system at which the energy balance arises additionally limits the parameters of dissipative solitons. This fact speaks of the enhanced stability of such solitons, and makes them attractive features for various applications in optoelectronics and nanoelectronics [43].

The Fourier spectra, at three different time instants, for the Bessel pulse amplitude are shown in Fig. 6.

Figure 6 shows that first the spectrum is enriched due to interaction with the medium, which has a variable refractive index, and then the efficient generation of higher harmonics occurs. Note that it has a non-negligible output even for harmonics above the third one. This makes such systems promising in devices for generating pulses with a wide spectrum.

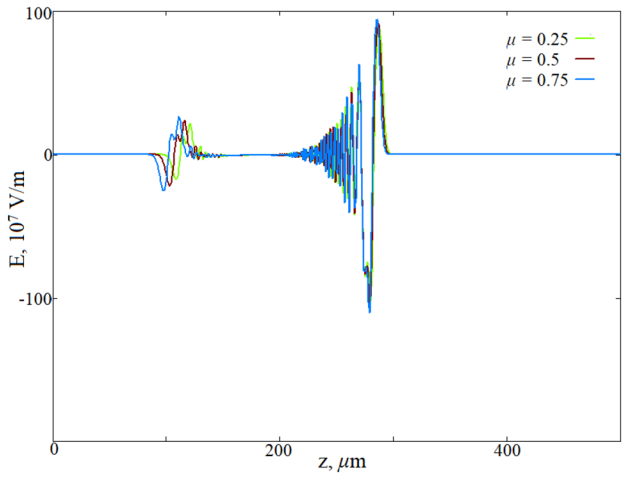


**Fig. 7** Longitudinal sections of a three-dimensional diffractionless ESOP with a Bessel cross-section in an optically anisotropic photonic crystal made of CNTs at a time instant of 15 ps and at $r = 0$ for different refractive index modulation depths: $\mu = 0.25$, $\mu = 0.5$, and $\mu = 0.75$

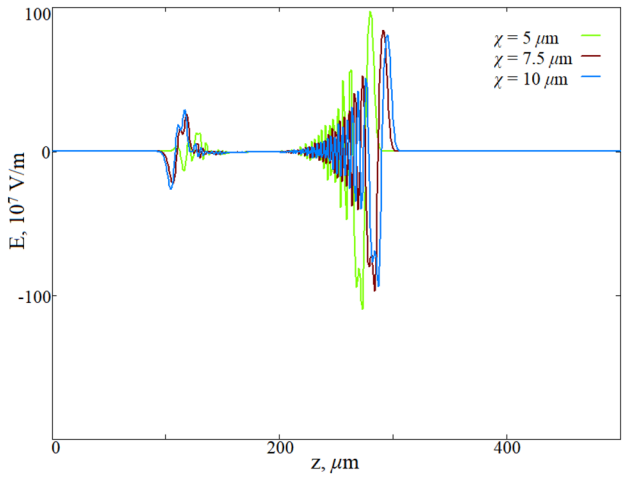


**Fig. 8** Longitudinal sections of a three-dimensional diffractionless ESOP with a Bessel cross-section in an optically anisotropic photonic crystal made of CNTs at a time instant of 15 ps and at $r = 0$ for different refractive index modulation depths: $\chi = 5\,\mu$m, $\chi = 7.5\,\mu$m, and $\chi = 10\,\mu$m

The next two Figs. 7 and 8 show the influence of the modulation parameters of the refractive index of an optically anisotropic photonic crystal based on CNTs on the group velocity of the pulse wave packet and on the shape of its envelope.

The parameters of the refractive index of an optically anisotropic photonic CNT crystal have little effect on the group velocity and the shape of the envelope of the ESOP with a Bessel cross-section. An increase in the modulation period of the refractive index leads to an increase in the pulse velocity. This is due to the fact that the pulse less often collides with the lattice sites of the photonic crystal, and, accordingly, is less often reflected from them, and therefore the interference processes also occur less frequently. The results presented in Figs. 7 and 8 have been repeatedly confirmed by other studies of the propagation of ESOP in photonic crystals and Bragg media [21–23].

## 4 Conclusions

As a result of the study of the dynamics of a three-dimensional diffractionless ESOP with a Bessel cross-section in an optically anisotropic photonic crystal based on semiconductor CNTs under the action of an external pumping field and the presence of nonlinear absorption, the following conclusions can be drawn:

(i) A three-dimensional ESOP with a Bessel cross-section, stably propagates in an optically anisotropic photonic crystal formed by CNTs, conserving its energy in a limited spatial region.
(ii) When changing the initial conditions, the shape of the pulse front does not change, thus dissipative solitons are revealed in the medium.
(iii) The modulation of the refractive index of an anisotropic photonic crystal affects the group velocity as well as the shape of the ESOP with a Bessel cross-section.

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