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RESEARCH ARTICLE

Ensuring safe robust control design through optimal control with switching strategy

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Abstract

This paper presents the design of a safe- robust control law for a class of nonlinear systems subjected to model and input uncertainties. Safety is addressed by means of control barrier functions. The design method involves defining an over-approximate region of the safe set and using it as a constraint. We then convert this safe- robust control problem into a constrained-optimal control problem for a nominal system with a modified cost-functional. Subsequently, this constrained-optimal control problem is reformulated as a minimization of a Hamilton-Jacobi-Bellman equation subjected to safety constraints, which is solved using the Karush-Kuhn-Tucker multiplier method. The design of the control law is facilitated by the use of a switching function when moving from safe to unsafe regions. This approach is termed the modular design approach, which maintains the system's robustness while preserving safety. Finally, numerical simulations are carried out to validate the proposed controller and test its efficacy on nonlinear systems with uncertainties.

KEYWORDS

control barrier functions, Hamilton-Jacobi-Bellman equations, safe robust control, uncertain systems

1 | INTRODUCTION

With the rising interest in cyber-physical systems, robotics, networked control systems, etc., the concept of safety has become an integral part of the controller's design [3, 4, 31]. Informally, safety is linked to preventing unexpected and/or undesirable events or outcomes. However, in a formal context, it pertains to the idea of maintaining set invariance. Invariance in this context means that any trajectory starting from the invariant set will never reach its complement set, thus ensuring the safety guarantee. Nagumo utilized the concept of the tangent cone to prove the set invariance of a set discussed in [7, 18]. Barrier certificates were introduced [22] to guarantee safety for nonlinear systems. Recently, Lyapunov-like functions the so-called barrier functions have emerged as a tool to establish safety guarantees; the primary goal of barrier functions is to avoid undesirable or unsafe regions. Control barrier functions (CBFs) can be seen as the extension of the general concept of barrier functions applied to a system involving a control input [3, 4]. The notion of CBF is defined with respect to a safe set, which is positive inside the safe set, zero at its boundary, and negative outside the safe set [27]. CBFs can be classified as zeroing CBFs and reciprocal CBFs [4]. However, in this paper, we specifically use zeroing CBFs due to its advantage in handling model uncertainties [33], thereby leading to a robust controller design.

The idea of CBFs has been explored to design the appropriate control law in relation to the system's safety [32]. Similar to ensuring safety, the tool to address systems' stability often relies on the concept of control Lyapunov functions (CLFs) [12, 28]. With the increasing interest in multi-objective control, CBF has gained attention due to its ability to combine with the CLFs and render the set invariant [26]. Although safety is a critical aspect of a system, it is not the only one, and performance can naturally be considered an equally important criterion. Thereby, combining the objective of optimality with safety leads to the key concept of safe optimal control. Specifically, *safe optimal control problems* optimize the objective function while concurrently incorporating safety as a constraint. CBFs have been extensively used to design safe optimal control laws. Though it is crucial to deal with constrained optimal control problems, when the inputs involved are affine, and the cost is quadratic, the problem reduces to solving a quadratic programming problem [4, 10]. Various research works have been reported dealing with safe optimal control; for instance, in [9], the dual relationship between the value and density functions is used to establish an optimal safe controller. Recently, [2] proposed an optimal controller encoded with safety constraints by formulating a constrained optimization problem.

In practice, with the increasing complexity of systems, it becomes onerous, if not outright impossible, to find the exact model of a system. Moreover, no model is entirely error-free and contains uncertainties broadly categorized as matched (i.e., uncertainty is in the range space of the input matrix) and mismatched [21]. In addition to model uncertainty, external disturbances and time-varying parameters are other primary sources of uncertainty. Therefore, various techniques have been used, such as the optimal control approach [12], H_∞/H_2 approach [11, 14] and the parametric approach [5, 38] to design a robust control law. In particular, F. Lin has extended the use of optimal control problems for a nominal system to design a robust control law in the presence of matched and mismatched uncertainties [16]. This control framework has been extensively applied along with various techniques such as event-triggered control [29], learning-based control [1, 30], etc. Recent studies have made significant progress in learning and critic-based control of nonlinear systems, including adaptive dynamic programming [34, 36, 37], reinforcement learning [23], and event-triggered mechanisms [24, 35] for handling safety and input constraints. In particular, barrier-critic adaptive robust control [25] has been proposed for state-constrained nonzero-sum differential games under uncertainty. In [19], a robust quadratic program is formulated to ensure stability and safety via robust constraints, while [17] adopts an adaptive strategy to jointly address safety and robustness under model uncertainty. Notably, both [19] and [17] enforce safety under uncertainty by modifying the safety condition, which leads to tightening of the safety condition.

Nonetheless, most existing works focus on robust control design without explicitly addressing safety, or enforce safety through quadratic programming-based approaches, while safety-oriented optimal control formulations such as [2] do not consider system uncertainties. Moreover, several safety-critical control methods handle uncertainty by tightening safety constraints, which can limit their applicability under uncertainty. This motivates to design a control law that jointly integrates safety and robustness for a class of nonlinear systems with matched uncertainty within the optimal control framework. This is achieved through converting a safe, robust control problem into a safe optimal control one for a nominal system with a modified cost-functional accounting for uncertainty, subjected to a safety constraint. Additionally, we exploit the concept of a switching function to select the control law to ensure safety and robustness depending on the value of the safety constraint. To accommodate the switching function into the safety constraint, we introduce an over-approximate region of the safe set. The control law consists of two components, one responsible for ensuring safety and the other for robust stability.

1.1 | Contributions

This paper presents a safe- robust controller design framework in the presence of input and model uncertainties. Our contributions are the following:

- A safe- robust control law is proposed for a nonlinear system considering a predefined family of closed sets. The control law design is based on reformulating the safe robust control problem into a constrained optimal control problem for a nominal system with a modified cost-functional.
- Leveraging the concept of switching, we introduce a switching function for a smooth transition between the control laws depending on the safety requirements. This results in a modular approach, where robustness and safety are assured independently.
- The proposed safe robust controller offers safety and stability guarantees in the presence of uncertainties.

1.2 | Notations

Throughout this paper, R and R^+ denote the set of real numbers and positive real numbers, respectively, and R^n denotes the n-dimensional real vector space. $\mathcal{D} \subset R^n$ denotes the set of state space, and $\mathcal{U}$ denotes the set of admissible control inputs. Int(C) indicates the interior region of a closed set C and ∂C denotes its boundary. The transpose of a matrix b is denoted by b^T. The Euclidean norm is denoted by $\|\cdot\|$. A function f: $R^n \to R^m$ is said to be Lipschitz continuous if there exists a constant $L \in R^+$ such that $\|f(x_2) - f(x_1)\| \leq L\|x_2 - x_1\| \ \ \forall x_1, x_2 \in R^+$. A continuous function $\alpha : [0; a) \to [0; \infty)$ with $a > 0$ is said to belong to the class κ, if $\alpha(0) = 0$ and if it is strictly increasing.

2 | PRELIMINARIES

This section provides the basic definitions and theorems required to derive the main results. Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x(t) \in \mathcal{D}$ and $u(t) \in \mathcal{U}$ are the state and control input, respectively. Here, the functions $f : R^n \to R^n$ and $g : R^n \to R^{n\times m}$ are locally Lipschitz continuous nonlinear functions. To build upon the concept of safety, we introduce the notion of CBF in the next subsection.

2.1 | Control barrier function (CBF)

The concept of barrier function has evolved to enable the enforcement of safety, which led to the introduction of CBF. Further discussions involve the following definitions and theorems presented in [2–4].

Definition 1 (Forward Invariance [3]). A set $C \subset \mathbb{R}^n$ is forward invariant with respect to system (1) if its solution starting from any $x(t_0) \in C$ stays within C for all future times, that is, $x(t) \in C, \ \forall t \geq t_0$.

Definition 2 (CBF). A continuously differentiable function $h : R^n \to R$ is a CBF for set C defined for (1), if there exists a class κ function 'σ' such that,

$$\sup_{u\in\mathcal{U}} \left[\frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x)u + \sigma(h(x)) \right] \geq 0, \quad x \in C. \tag{2}$$

Let $\sigma(h(x)) = \alpha h(x)$ with $\alpha > 0$.

Theorem 1. *Consider a set C defined as $C = \{x : h(x) \geq 0\}$. If h is a CBF on $\mathcal{D}$ and $\frac{\partial h(x)}{\partial x} \neq 0, \ \forall x \in \partial C$, then any Lipschitz continuous controller $u(x(t)) = K(x(t))$ for (1) satisfying (2) renders the set C forward invariant.*

As shown in [7], if C is rendered forward invariant, then the system is safe with respect to the set C. Here, C is referred to as the safety set. It is important to note that the theorem's conclusion not only guarantees safety but also ensures stability when combined with CLFs. The reader is referred to [3] for more details. Next, we discuss the concept of robust control framework as introduced in [16].

2.2 | Optimal control approach for robust control design

The system (1) does not involve any uncertain parameters, whereas all practical control systems face errors or disturbances, inevitably leading to uncertainty. Thus, a more general description accounting for uncertainty is

$$\dot{x} = f(x) + \Delta f(x) + g(x)u, \tag{3}$$

where $\Delta f(x)$ represents bounded uncertainty present in the system and (1) constitutes the nominal model for (3). The bound on the uncertainty $\Delta f(x)$ is defined as

$$\|\Delta f(x)\| \leq f_{\max}, \tag{4}$$

where $f_{\max}(x)$ is a nonnegative function. To deal with uncertainty $\Delta f(x)$, F. Lin presented an optimal control approach for designing a robust control law [15]. The following cost-functional is proposed and is referred to as a modified cost due to the inclusion of $f_{\max}$

$$V(x) = \int_0^\infty F(f_{\max}, x, u)\, dt. \tag{5}$$

The underlying concept introduced in [15] is discussed in the following lemma.

Lemma 1. *With bounded system uncertainty $\Delta f(x)$, the solution of the optimal control problem for the nominal system (1), which minimizes the modified cost-functional (5) is a robust solution of the uncertain system (3).*

Proof. The detailed proof of this lemma can be obtained from [15]. ■

3 | PROBLEM DESCRIPTION AND PROPOSED SOLUTION

3.1 | Problem description

Uncertainty can enter the system through various channels. The system dynamics (3) contains uncertainty in $f(x)$. Suppose the uncertainty is present in $f(x)$ as well as in $g(x)$. Then, the system dynamics can be described as

$$\dot{x} = f(x) + \Delta f(x) + (g(x) + \Delta g(x))u, \tag{6}$$

where $x(t) \in \mathcal{D}$ and $u(t) \in \mathcal{U}$ are the state and control input. We assume the system is stabilizable. Let us consider that the uncertainties $\Delta f(x)$ and $\Delta g(x)$ satisfy the matching condition [21], that is, $\Delta f(x)$ and $\Delta g(x)$ lie in the range space of the input matrix $g(x)$. Specifically, we consider $\Delta f(x) = g(x)\phi q(x)$, which represents the state uncertainty, and $\Delta g(x) = g(x)\phi r(x)$ represents the uncertainty in the input matrix. Here, ϕ is a known scaling matrix and $q(x)$, $r(x)$ are unknown functions. Hence, (6) can be recast as

$$\dot{x} = f(x) + g(x)\phi q(x) + (g(x) + g(x)\phi r(x))u. \tag{7}$$

System (7) satisfies the following assumptions:

Assumption 1. $f(0) = 0$ and $q(0) = 0$, so that $x = 0$ is the equilibrium point.

Assumption 2. $q(x)$ is bounded, which means that $\|\epsilon^{-\frac{1}{2}}\phi q(x)\| \le q_{\max}(x)$, where $q_{\max}(x)$ is a nonnegative function and $\epsilon \in (0, 1)$ is a suitable scalar.

Assumption 3. The function $r(x)$ is bounded according to the following inequality $\|r(x)\| \le \frac{2}{\|\phi\|}$.

The objective is to attain stability with respect to the equilibrium point, even in the presence of uncertainty. However, in practical scenarios, there is a need to balance stability and safety requirements. Consequently, the challenge is to drive the system to the equilibrium point while avoiding the unsafe region in the presence of uncertainty. The notion of safety is defined with respect to forward invariance (Theorem 1) of the safe set $\mathcal{C}$ for the nonlinear system (7). Thus, the state space $\mathcal{D}$ is divided into a safe region defined as $\mathcal{C} = \{x : h(x) \ge 0\}$ and an unsafe region, $\overline{\mathcal{C}} = \{x : h(x) < 0\}$. The equilibrium point is assumed to lie strictly within the safe set, ensuring compatibility between safety and stability objectives. Thus, the overall control problem can be formulated as

Problem 1. Design a state feedback control law u_{safe} that ensures the safety and stability of system (7) under the presence of bounded uncertainty satisfying Assumptions 2 and 3. In particular, the control law should be formulated to ensure safety when the system approaches an unsafe region while maintaining the flexibility to address cases where safety is not the primary objective.

Building upon the problem described here, the next subsection provides insight into the approach adopted for the design of the controller.

3.2 | Overview of the proposed solution

Figure 1 illustrates the steps used to design the proposed control law. The block diagram consists of three primary blocks. First, we describe the system dynamics based on the influence of uncertainty. Second, we address the safety criteria by introducing a CBF and the associated safety constraints. Next, to design the safe robust control law

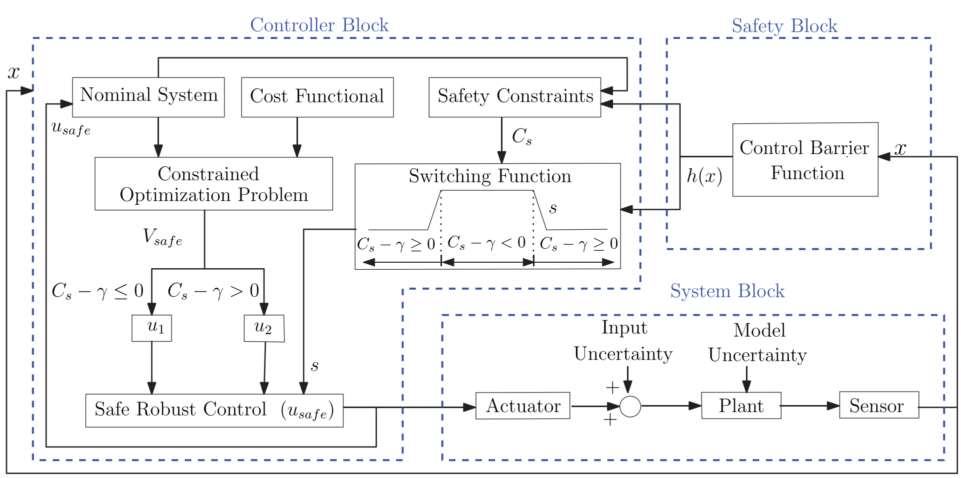


FIGURE 1 Block diagram for proposed safe robust control scheme.

mentioned in Problem 1, we consider the optimal control approach presented in [15]. However, our goal of designing a safe robust control law cannot be directly achieved through Lemma 1 due to the involvement of safety considerations. Hence, the problem is reformulated as solving a safe optimal control problem, depicted as a controller block in Figure 1, for the nominal system

$$\dot{x} = f(x(t)) + g(x(t))u, \quad x(0) = x_0, \tag{8}$$

constrained by the following inequality

$$C_s := \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x)u + \alpha h(x) \geq 0. \tag{9}$$

The cost functional associated with solving the safe optimal control problem is given as

$$V(x,u) = \int_0^{\infty} (\lambda \beta(x)^2 + q_{\max}(x)^2 + x^T x + u^T u)dt, \tag{10}$$

where $\beta(x)^2 = \beta_1(x)^2 + \beta_2(x)^2$ such that

$$\beta_1(x) = \frac{s(x)}{h(x)}, \quad \beta_2(x) \geq \left\| s(x)\frac{\partial h(x)}{\partial x} g(x) \left[\frac{1}{1-\epsilon} I\right]^{\frac{1}{2}} \right\|, \quad \lambda \geq 0. \tag{11}$$

Here, λ is a positive scalar and $s(x)$ is a switching function with $s(x) \in [0, 1]$ which are discussed in Section 4. It is important to emphasize that the cost defined in (10) differs from the cost introduced in [15]. Specifically, in (10), we include an additional term $\lambda \beta(x)^2$ that accounts for the presence of the safety constraint C_s. Moreover, for a predefined CBF, $h(x)$, the safety of (6) is ensured using the term $\lambda \beta(x)^2$. It should be noted that $\beta(x)$ is designed such that it flexibly modifies (10) depending on the switching mechanism, discussed in Section 4. Furthermore, $q_{\max}$ is introduced in the cost (10) to accommodate the bound on uncertainty. Thus, the problem for the nominal system (8) can be formulated as solving the constrained optimal control problem when there is a risk of entering the unsafe region by minimizing the following generalized Hamilton-Jacobi-Bellman (HJB) equation

$$\min_{u \in \mathcal{U}} \left[\lambda \beta(x)^2 + q_{\max}(x)^2 + x^T x + u^T u + \left[\frac{\partial V(x)}{\partial x}\right]^T f(x) + \left[\frac{\partial V(x)}{\partial x}\right]^T g(x)u \right] = 0, \tag{12}$$

subjected to the safety constraint (9) and solving the unconstrained problem elsewhere. Let u_1 and u_2 denote the control law obtained by solving the constrained and the unconstrained optimization problem, respectively. In order to have a smooth transition between the control inputs, a switching function $s(x)$ is designed. This inclusion of the switching function modifies the safety constraint to $C_s - \gamma$, depicted in the controller block of Figure 1. Here, γ is a small positive scalar whose design is considered in the next section. Thus, the proposed safe robust control law can be represented as

$$u_{\text{safe}} = s(x)u_1 + (1 - s(x))u_2. \tag{13}$$

The next section provides a comprehensive discussion of the design of $s(x)$ and the steps to formulate the control law (13). Additionally, with a suitable selection of Lyapunov function, we further analyze the stability of system (7).

4 | MAIN RESULTS

This section is organized into four distinct subsections. We begin by examining the design of the switching function, followed by a detailed discussion of the steps involved in the controller design. Subsequently, a stability analysis is conducted, and finally, we extend the results of nonlinear systems to linear systems as a special case.

4.1 | Design of switching function

The design of the switching function is motivated by the idea of avoiding the infinitely fast switching between the controllers u_1 and u_2. We first introduce the following differentiable switching function

$$s(x) = \begin{cases} 1 & \text{if } C_s < 0, \\ 3\left(\frac{h(x)}{\gamma}\right)^4 - 4\left(\frac{h(x)}{\gamma}\right)^3 + 1 & \text{if } 0 \leq C_s \leq \gamma, \\ 0 & \text{if } C_s > \gamma, \end{cases} \tag{14}$$

where γ is a small positive constant. The switching function (14) generates three regions: an unsafe region, a transition phase, and a safe region. To incorporate this switching structure into the optimization framework, the safety constraint is modified from $C_s \leq 0$ to $(C_s - \gamma) \leq 0$. Additionally, this modified safety constraint leads to the introduction of $h'(x)$, which is the over-approximate region of $h(x)$ defined as

$$h'(x) = h(x) - \rho'^2, \tag{15}$$

where ρ' is a small positive constant. The relation between ρ' and γ can be established using (2) and Theorem 1, which outlines the correlation between the safe set and safety constraint. Hence, the safety constraint (9) with $h'(x)$ as the CBF can be expressed as

$$\frac{\partial h'(x)}{\partial x}f(x)+\frac{\partial h'(x)}{\partial x}g(x)u+\alpha h'(x)\geq 0. \tag{16}$$

Substituting the value of $h'(x)$ using (15), we obtain

$$\frac{\partial h'(x)}{\partial x}f(x)+\frac{\partial h'(x)}{\partial x}g(x)u+\alpha(h(x)-\rho'^2)\geq 0. \tag{17}$$

Also, the partial derivative of (15) yields $\frac{\partial h'(x)}{\partial x}=\frac{\partial h(x)}{\partial x}$. Hence, (17) can be rewritten as

$$\frac{\partial h(x)}{\partial x}f(x)+\frac{\partial h(x)}{\partial x}g(x)u+\alpha h(x)-\alpha\rho'^2\geq 0, \tag{18}$$

with $\gamma=\alpha\rho'^2$. Considering (9), (18) can be viewed as $\hat{C}_s=C_s(x)-\gamma$ where $\hat{C}_s$ represents the modified safety constraint (16) associated with the safe set $h'(x)$. This modified safety constraint is referred to as the safety constraint and is further used to derive the results presented in the following subsections.

Remark 1. In the proposed framework, the switching function $s(x)$ is restricted to be continuous and differentiable, satisfying $s(x)\in[0,1]$ for all x. The quartic polynomial in (14) ensures that the resulting control law u_{safe} remains continuous and bounded. Thus, the smooth transition between u_1 and u_2 prevents the abrupt changes in the control input and mitigates chattering effects near the safety boundary while preserving stability and safety guarantees.

4.2 | Controller design

The design steps of the control law are discussed in two parts. Initially, we address the case when the safety constraint is active, that is, $\hat{C}_s\leq 0$, and solve the constrained optimal control problem. Next, we assume the safety constraint is inactive and solve the optimal control problem without constraints. The approach to design the control law is based on solving the constrained optimal control problem employing Karush-Kuhn-Tucker (KKT) optimality conditions [8] and using the unconstrained solution elsewhere. Note that the switching function $s(x)$ is introduced along with $\hat{C}_s$, thus transforming the safety constraint to $s\hat{C}_s$. As s is a positive constant, it does not affect the convexity of the safety constraint [20]. To solve the outlined problem, the Lagrangian function E can be constructed using (12) and (16) taken along with $s(x)$ as

$$\begin{aligned}E=\;&\lambda\beta(x)^2+q_{\max}(x)^2+x^Tx+u^Tu+\left[\frac{\partial V}{\partial x}\right]^Tf(x)\\&+\left[\frac{\partial V}{\partial x}\right]^T g(x)u-k^Ts(x)\\&\left[\frac{\partial h'(x)}{\partial x}f(x)+\frac{\partial h'(x)}{\partial x}g(x)u+\alpha h'(x)\right].\end{aligned} \tag{19}$$

Applying the KKT conditions, $\left[\frac{\partial E}{\partial u}\right]_{u_1}=0$, $k^Ts(x)\hat{C}_s=0$ and $k\geq 0$ on (19), we obtain

$$2u_1^T+\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^Tg(x)-k^Ts(x)\left[\frac{\partial h'(x)}{\partial x}g(x)\right]=0, \tag{20}$$

$$k^Ts(x)\left[\frac{\partial h'(x)}{\partial x}f(x)+\frac{\partial h'(x)}{\partial x}g(x)u_1+\alpha h'(x)\right]=0, \tag{21}$$

$$k\geq 0, \tag{22}$$

where V_{safe} denote the corresponding value function obtained by solving the following HJB

$$\begin{aligned}&\lambda\beta(x)^2+q_{\max}(x)^2+x^Tx+\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^Tf(x)\\&-\frac{1}{2}\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^Tg(x)g(x)^T\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]\\&+\frac{1}{4}k^Ts(x)\left[\frac{\partial h'(x)}{\partial x}\right]g(x)g(x)^T\left[\frac{\partial h'(x)}{\partial x}\right]^Ts(x)k=0.\end{aligned} \tag{23}$$

The detailed steps to solve (23) numerically are presented in the supplementary material. It should be noted that if $s(x)\hat{C}_s\leq 0$, k is positive and it is zero otherwise. By solving (20), (21), and (22) simultaneously, we obtain the following expression for the control law

$$u_1=-\frac{1}{2}\left[g(x)^T\frac{\partial V_{\text{safe}}}{\partial x}-g(x)^T\frac{\partial h'(x)}{\partial x}^Ts(x)k\right], \tag{24}$$

when $\hat{C}_s\leq 0$ with the KKT multiplier as

$$k=\frac{-\frac{\partial h'(x)}{\partial x}f(x)+\frac{1}{2}\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial V_{\text{safe}}}{\partial x}-\alpha h'(x)}{\frac{1}{2}s\left[\frac{\partial h'(x)}{\partial x}\right]g(x)g(x)^T\left[\frac{\partial h'(x)}{\partial x}\right]^T}, \tag{25}$$

and when $\hat{C}_s$ is inactive, (24) reduces to

$$u_2=-\frac{1}{2}g(x)^T\frac{\partial V_{\text{safe}}}{\partial x}, \tag{26}$$

Algorithm 1. Safe Robust Control Design

Given: System dynamics f, g ;
Cost parameters $q_{\max}, \beta, \lambda$;
Safety parameters h, α, γ ;
Define: Switching Function s and $h'(x)$;
Initialize u_0, t_0, x_0, t_f $\triangleright$ t_f denotes the final time
for $i = 1 : t_f$ **do**
  Calculate: $C_s(i)$ from (9).
  if $C_s(i) \leq 0$ **then**
    Put $s(i) = 1$
  else if $0 < C_s(i) < \gamma$ **then**

$$s(i) = 3\left(\frac{h(x)}{\gamma}\right)^4 - 4\left(\frac{h(x)}{\gamma}\right)^3 + 1$$

  else $C_s \geq \gamma$
    Put $s(i) = 0$
  end if
  Calculate $V_{\text{safe}}(i)$ and k from (23) and (25) respectively.
  Return $u_{\text{safe}}(i)$
end for

for which V_{safe} can be obtained by solving the following HJB equation

$$\lambda\beta(x)^2 + q_{\max}(x)^2 + x^T x + \left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T f(x) - \frac{1}{2}\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T g(x)g(x)^T\left[\frac{\partial V_{\text{safe}}}{\partial x}\right] = 0. \quad (27)$$

With the help of the switching function (14), the control laws (24) and (26) can be combined and expressed as (13). The design of such a safe- robust control law allows for dealing with safety and robustness separately, leading to a modular design. The steps to design the control law u_{safe} are discussed in Algorithm 1.

Remark 2. For a single input system with safety constraint (16), the expression for u_1 reduces to

$$u_1 = -g(x)^{-1}f(x) - g(x)^{-1}\frac{\partial h'(x)}{\partial x}^{-1}\alpha h'(x).$$

Interestingly, the control law depends on the parameter α, which is chosen arbitrarily. Moreover, solving the modified HJB equation is not necessary for such cases when the constraint is active, as the equation simplifies to an expression that does not contain the solution of the HJB equation (23). However, the HJB equation (27) is taken into account while solving for $\hat{C}_s > 0$.

4.3 | Computation of u_1 and u_2 using safe successive approximation

This section describes the methodology to solve the optimization problem for both scenarios when the safety constraint is active and when it is inactive. The method builds on the successive Galerkin approximation technique of Beard [6], later extended as safe Galerkin successive approximation (SGSA) in [2]. In this work, the SGSA method is applied with a modified cost functional. We first consider the case in which the safety constraint is active, and rewrite the HJB equation (23) as

$$Q(x) + \left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T f(x) - \frac{1}{2}\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T g(x)g(x)^T\left[\frac{\partial V_{\text{safe}}}{\partial x}\right] + \frac{1}{4}k^T s\left[\frac{\partial h'(x)}{\partial x}\right]g(x)g(x)^T\left[\frac{\partial h'(x)}{\partial x}\right]^T sk = 0, \quad (28)$$

where $Q(x) = \lambda\beta(x)^2 + q_{\max}(x)^2 + x^T x$ and the KKT multiplier is given by (25). For notational simplicity considering $\mathbb{M} = \frac{1}{2}s\left[\frac{\partial h'(x)}{\partial x}\right]g(x)g(x)^T\left[\frac{\partial h'(x)}{\partial x}\right]^T$. Thereby, (25) becomes

$$k = -\mathbb{M}^{-1}\frac{\partial h'(x)}{\partial x}f(x) + \frac{1}{2}\mathbb{M}^{-1}\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial V_{\text{safe}}}{\partial x} - \mathbb{M}^{-1}\alpha h'(x). \quad (29)$$

Using (29), the control law (24) can be modified as

$$u_1 = -\frac{1}{2}g(x)^T\frac{\partial V_{\text{safe}}}{\partial x} + \frac{1}{4}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)\,\mathbb{M}^{-1}\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial V_{\text{safe}}}{\partial x} + \frac{1}{2}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)\left[-\mathbb{M}^{-1}\frac{\partial h'(x)}{\partial x}f(x) - \mathbb{M}^{-1}\alpha h'(x)\right]. \quad (30)$$

Let u_1 be the controller on a compact set Ω. To compute the control law u_1, we assume $V_{\text{safe}} = \sum_{j=1}^{N} c_j\psi_j$, where ψ_j is the j^{th} basis function and c_j is the j^{th} constant coefficient and thus $\frac{\partial V_{\text{safe}}}{\partial x} = \sum_{j=1}^{N} c_j\frac{\partial\psi_j}{\partial x}$. In order to simplify notation, let $\boldsymbol{\psi_N} \triangleq (\psi_1, \ldots\ldots, \psi_N)^T$, $\boldsymbol{c_N} \triangleq (c_1(i), \ldots\ldots, c_N(i))^T$, $\Delta\boldsymbol{\psi_N} \triangleq \left(\frac{\partial\psi_1}{\partial x}, \ldots\ldots, \frac{\partial\psi_N}{\partial x}\right)^T$. Given a set of polynomial $\boldsymbol{\psi_N}$, based on Section 2.3.1 of [6], the generalized HJB (12) can be simplified and approximated as

$$(A_1 + A_2(u_1))\boldsymbol{c_N} = b_1 + b_2(u_1),$$

where $A_1 = \int_\Omega \boldsymbol{\psi_N} f(x)^T\Delta\boldsymbol{\psi_N^T}dx$, $A_2(u_1) = \int_\Omega \boldsymbol{\psi_N} u_1^T g^T \Delta\boldsymbol{\psi_N^T}dx$, $b_1 = -\int_\Omega Q(x)\boldsymbol{\psi_N}dx$, $b_2(u_1) = -\int_\Omega \left(u_1^T u_1\right)\boldsymbol{\psi_N}dx$. Note that A_1 and b_1 are computed once, but for iterative approximations of u_1, $A_2(u_1)$ and $b_2(u_1)$ must be recalculated in each run. This emphasizes the necessity of simplifying these expressions. Thus, by utilizing gradient of V_{safe} and substituting it in (30), we obtain $u_1 = -u_{\text{com1}}\Delta\boldsymbol{\psi}_{\boldsymbol{N}}^{\boldsymbol{T}}\boldsymbol{c}_{\boldsymbol{N}} + u_{\text{com2}}$, with $u_{\text{com1}} = \frac{1}{2}g(x)^T - \frac{1}{4}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)\mathbb{M}^{-1}\frac{\partial h'(x)}{\partial x}g(x)g(x)^T$, $u_{\text{com2}} = \frac{1}{2}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)\left[-\mathbb{M}^{-1}\frac{\partial h'(x)}{\partial x}f(x) - \mathbb{M}^{-1}\alpha h'(x)\right]$. Hence, $A_2(u_1)$ and $b_2(u_1)$ can be computed as

$$A_2(u_1) = -\sum_{j=1}^{N} c_j\mathbb{H}_{1j} + \mathbb{H}_2,$$

$$b_2(u_1) = -\sum_{j=1}^{N} c_j(\mathbb{H}_{3j}\boldsymbol{c}_{\boldsymbol{N}} + \mathbb{H}_{4j}) - \mathbb{H}_5,$$

where $\mathbb{H}_{1j} = \int_\Omega \boldsymbol{\psi}_{\boldsymbol{N}}\frac{\partial\psi_j}{\partial x}^T u_{\text{com1}}^T g^T\Delta\boldsymbol{\psi}_{\boldsymbol{N}}^T dx$, $\mathbb{H}_2 = \int_\Omega \boldsymbol{\psi}_{\boldsymbol{N}} u_{\text{com2}}^T g^T\Delta\boldsymbol{\psi}_{\boldsymbol{N}}^T dx$, $\mathbb{H}_{3j} = \int_\Omega \boldsymbol{\psi}_{\boldsymbol{N}}\frac{\partial\psi_j}{\partial x}^T u_{\text{com1}}^T u_{\text{com1}}\Delta\boldsymbol{\psi}_{\boldsymbol{N}} dx$, $\mathbb{H}_{4j} = \int_\Omega 2\boldsymbol{\psi}_{\boldsymbol{N}}\frac{\partial\psi_j}{\partial x}^T u_{\text{com1}}u_{\text{com2}}dx$, and $\mathbb{H}_5 = \int_\Omega \boldsymbol{\psi}_{\boldsymbol{N}} u_{\text{com2}}^T u_{\text{com2}}dx$. Similar steps can be followed to obtain the control law u_2 by substituting $k = 0$. This section presents the essential steps for computing u_1 and u_2; detailed derivations and the algorithm are included in the supplementary material owing to space limitations.

4.4 | Stability analysis

The following theorem proves the stability of the proposed control law (13). Before presenting the main theorem, a preliminary lemma inspired by [13] is provided.

Lemma 2. *Let* $\frac{1}{2}g(x)^T\frac{\partial h'(x)}{\partial x}^T sk$ *and* $\phi q(x)$ *be the matrices of suitable dimensions. If there exist a positive scalar* ϵ *satisfying* $\epsilon^{-1}I - I > 0$, $\epsilon > 0$; *then*

$$\begin{aligned} &q(x)^T\phi^T\phi q(x) + \frac{1}{2}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\phi q(x) \\ &\quad + \frac{1}{2}q(x)^T\phi^T g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k \\ &\leq \frac{1}{4}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)(\epsilon^{-1}I - I)^{-1}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k \\ &\quad + \epsilon^{-1}q(x)^T\phi^T\phi q(x). \end{aligned} \tag{31}$$

Proof. See Appendix A. ∎

Theorem 2. *For the KKT multiplier defined in (25) and a positive scalar* λ*, consider the system dynamics (7) with uncertainties verifying Assumptions 2 and 3. The safe robust control law* u_{safe} *defined in (13) is safe and asymptotically stable for the uncertain system (7) if the inequality (11) along with the following condition is satisfied*

$$\lambda \geq \frac{k^T k}{4}. \tag{32}$$

Proof. We first show that the origin of (7) is asymptotically stable under the control law u_{safe}. To prove the stability, we consider V_{safe} obtained from (23) as the Lyapunov function. Hence, $\dot{V}_{\text{safe}}$ for (7) is computed as

$$\begin{aligned} \dot{V}_{\text{safe}}(x) = &\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T [f(x) + g(x)u_{\text{safe}}] \\ &+ \left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T g(x)\phi q(x) \\ &+ \left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T g(x)\phi r(x)u_{\text{safe}}. \end{aligned} \tag{33}$$

Now, u_{safe} satisfies the HJB equation (12), therefore, from (12) and (20) we obtain

$$\begin{aligned} &\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T [f(x) + g(x)u_{\text{safe}}] = -\lambda\beta(x)^2 \\ &\quad - q_{\max}(x)^2 - x^T x - u_{\text{safe}}^T u_{\text{safe}}, \end{aligned}$$

and $\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T g(x) = -2(u_{\text{safe}})^T + k^T s(x)\frac{\partial h'(x)}{\partial x} g(x)$. Substituting the above results in (33), we arrive at

$$\begin{aligned} \dot{V}_{\text{safe}}(x) = &-\lambda\beta(x)^2 - q_{\max}(x)^2 - x^T x - u_{\text{safe}}^T u_{\text{safe}} \\ &- 2u_{\text{safe}}^T\phi q(x) + k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\phi q(x) \\ &- 2u_{\text{safe}}^T\phi r(x)u_{\text{safe}} \\ &+ k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\phi r(x)u_{\text{safe}}. \end{aligned}$$

To further simplify the above equation, we add and subtract the following terms (i) $q(x)^T\phi^T\phi q(x)$, (ii) $\frac{1}{4}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k$ and (iii) $u_{\text{safe}}^T r(x)^T\phi^T\phi r(x)u_{\text{safe}}$.

$$\begin{aligned} &\dot{V}_{\text{safe}}(x) \\ &= -\lambda\beta(x)^2 - q_{\max}(x)^2 - x^T x - u_{\text{safe}}^T u_{\text{safe}} - 2u_{\text{safe}}^T\phi q(x) \\ &\quad + k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\phi q(x) - 2u_{\text{safe}}^T\phi r(x)u_{\text{safe}} \\ &\quad + k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\phi r(x)u_{\text{safe}} + q(x)^T\phi^T\phi q(x) \end{aligned}$$

$$
\begin{aligned}
&- q(x)^T\phi^T\phi q(x) + \frac{1}{4}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k\\
&- \frac{1}{4}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k\\
&+ u_{\text{safe}}^T r(x)^T\phi^T\phi r(x)u_{\text{safe}} - u_{\text{safe}}^T r(x)^T\phi^T\phi r(x)u_{\text{safe}}.
\end{aligned}
$$

Using the following relation $2u_{\text{safe}}^T\phi q(x) \le u_{\text{safe}}^T u_{\text{safe}} + q(x)^T\phi^T\phi q(x)$, $2u_{\text{safe}}^T\phi r(x)u_{\text{safe}} \le u_{\text{safe}}^T r(x)^T\phi^T\phi r(x)u_{\text{safe}} + u_{\text{safe}}^T u_{\text{safe}}$, and on clubbing the identical terms to form squared expressions, we obtain

$$
\begin{aligned}
&\dot V_{\text{safe}}(x)\\
&= -\lambda\beta(x)^2 - q_{\max}(x)^2 - x^T x - 2u_{\text{safe}}^T\phi r(x)u_{\text{safe}}\\
&\quad + q(x)^T\phi^T\phi q(x) + \frac{1}{2}k^T s\frac{\partial h'(x)}{\partial x}g(x)\phi q(x)\\
&\quad + \frac{1}{2}q(x)^T\phi^T g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k\\
&\quad + \frac{1}{4}k^T s\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k\\
&\quad + u_{\text{safe}}^T r(x)^T\phi^T\phi r(x)u_{\text{safe}}\\
&\quad - (u_{\text{safe}} + \phi q(x))^T(u_{\text{safe}} + \phi q(x))\\
&\quad - \left(\frac{1}{2}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k - \phi r(x)u_{\text{safe}}\right)^T\\
&\quad \left(\frac{1}{2}g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k - \phi r(x)u_{\text{safe}}\right).
\end{aligned}
$$

Following additional simplifications, that is, eliminating the last two negative terms, the above equality reduces to

$$
\begin{aligned}
&\dot V_{\text{safe}}(x)\\
&\le -\lambda\beta(x)^2 - q_{\max}(x)^2 - x^T x + (q(x)^T\phi^T\phi q(x)\\
&\quad + \frac{1}{2}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\phi q(x) + \frac{1}{2}q(x)^T\phi^T g(x)^T\\
&\quad \frac{\partial h'(x)}{\partial x}^T s(x)k) + \frac{1}{4}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)g(x)^T\\
&\quad \frac{\partial h'(x)}{\partial x}^T s(x)k - u_{\text{safe}}^T[2\phi r(x) - r(x)^T\phi^T\phi r(x)]u_{\text{safe}}.
\end{aligned}
$$

Applying Lemma 2 and substituting $[(\epsilon^{-1}I - I)^{-1} + I] = [\frac{1}{1-\epsilon}]I$ result in the following inequality for $\dot V_{\text{safe}}$

$$
\begin{aligned}
&\dot V_{\text{safe}}(x)\\
&\le -x^T x - q_{\max}(x)^2 + \epsilon^{-1}q(x)^T\phi^T\phi q(x)\\
&\quad - u_{\text{safe}}^T[2\phi r(x) - r(x)^T\phi^T\phi r(x)]u_{\text{safe}} - \lambda\beta(x)^2\\
&\quad + \frac{1}{4}k^T s(x)\frac{\partial h'(x)}{\partial x}g(x)\left[\frac{1}{1-\epsilon}I\right]g(x)^T\frac{\partial h'(x)}{\partial x}^T s(x)k.
\end{aligned}
\tag{34}
$$

Since $q(x)$ and $r(x)$ satisfy Assumptions 2, 3 and together with (11) if condition (32) holds, then the above inequality reduces to

$$
\dot V_{\text{safe}}(x) \le -x^T x. \tag{35}
$$

Thus, the Lyapunov conditions are satisfied, ensuring asymptotic stability. Next, the safety of system (7) can be established using the method of contradiction. Let us suppose that the initial state, $x_0 \in \text{Int}(C)$, and the set C is not forward invariant. Then, for a time instant t_1, $\exists\, x(t_1)$ such that $x(t_1) \to \partial C \;\Rightarrow\; h(x(t_1)) \to 0 \;\Rightarrow\; \beta_1(x(t_1)) \to \infty \;\Rightarrow\; V(x(t_1)) \to \infty$, which contradicts (35). Hence, we can conclude that C is forward invariant, and (7) is safe. Consequently, we can conclude that the proposed control law u_{safe} not only exhibits the asymptotic stability but also ensures safety. ■

Remark 3. When $s \ne 0$, that is, $\hat C_s \le 0$, (35) provides the stability guarantee, and the satisfaction of safety constraint assures safety. However, when $s = 0$, that is, $\hat C_s > 0$, the safety constraint remains inactive, leading to a simplification of (23) into a simplified HJB equation given as (27). In this context, (35) can be further simplified as $\dot V_{\text{safe}} \le -x^T x - \lambda\beta(x)^2$, which subsequently ensures the stability of the system (7).

Remark 4. The condition (32) in Theorem 2 is imposed as a Lyapunov verification criterion. Here, λ scales the safety-related term in cost (10), while $k(x)$ indicates when the safety constraint is active. Accordingly, λ is chosen to be sufficiently large to ensure a balance between safety enforcement and stabilization. Once this condition is verified over the compact domain Ω, the controller can be implemented without requiring any bounds of $k(x)$.

4.5 | Special case

The results obtained from the analysis of nonlinear systems can also be extended to the context of linear systems. Specifically, the functions $f(x)$ and $g(x)$ in (7) are considered linear functions and thus, the system dynamics can be expressed in a matrix form as

$$
\dot x = (A + \Delta A)x + (B + \Delta B)u, \tag{36}
$$

where A, B are known constant matrices and $\Delta A = B\phi q(p)$, $\Delta B = B\phi r(p)$ are the matched uncertainty present in the state and input, respectively, with p as the uncertain parameter. ϕ is a known scaling matrix and $q(p)$, $r(p)$ are unknown matrices satisfying $\|\epsilon^{-\frac{1}{2}}\phi q(p)\| \leq H$, $\|r(p)\| \leq \frac{2}{\|\phi\|}$ similar to Assumptions 2 and 3 respectively. To solve the optimal control problem, the nominal dynamics (1) and safety constraint (16) for the systems dynamics (36) are given as

$$\dot{x} = Ax + Bu, \tag{37}$$

$$C_s := \frac{\partial h'(x)}{\partial x}Ax + \frac{\partial h'(x)}{\partial x}Bu + \alpha h'(x) \geq 0. \tag{38}$$

The cost-functional (10) to solve the safe optimal control problem for (36) is given as

$$V(x, u) = \int_0^\infty (\lambda\beta(x)^2 + x^T Hx + x^T x + u^T u)dt, \tag{39}$$

where, λ is a positive scalar and $\beta(x)^2 = \beta_1(x)^2 + \beta_2(x)^2$ is such that

$$\beta_1(x) = \frac{s(x)}{h(x)}, \beta_2(x) \geq \left\| s(x)\frac{\partial h(x)}{\partial x}B\left[\frac{1}{1-\epsilon}I\right]^{\frac{1}{2}}\right\|, \lambda \geq 0. \tag{40}$$

Following the steps mentioned in Section 4, a safe robust control law can be designed for the nominal dynamics (37) and cost-functional (39) subjected to the safety constraint (38). As a result of the special case of Theorem 2 for linear systems, the following Corollary is introduced.

Corollary 1. *For a positive scalar λ, consider the dynamical system (36), the safe robust control law (13) with*

$$u_1 = -B^T\frac{\partial V_{\text{safe}}}{\partial x} + \frac{1}{2}B^T\frac{\partial h'(x)}{\partial x}^T k,$$
$$u_2 = -B^T\frac{\partial V_{\text{safe}}(x)}{\partial x},$$

where k is expressed as $k = \frac{-\frac{\partial h'(x)}{\partial x}Ax + \frac{1}{2}\frac{\partial h'(x)}{\partial x}BB^T\frac{\partial V_{\text{safe}}}{\partial x} - \alpha h'(x)}{\frac{1}{2}\left[\frac{\partial h'(x)}{\partial x}\right]BB^T\left[\frac{\partial h'(x)}{\partial x}\right]^T}$, and V_{safe} is the solution of the following HJB

$$\lambda\beta(x)^2 + x^T Hx + x^T x + \left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T Ax - \frac{1}{2}\left[\frac{\partial V_{\text{safe}}}{\partial x}\right]^T BB^T\left[\frac{\partial V_{\text{safe}}}{\partial x}\right] + \frac{1}{4}k^T s\left[\frac{\partial h'(x)}{\partial x}\right]BB^T\left[\frac{\partial h'(x)}{\partial x}\right]^T sk = 0, \tag{41}$$

is safe and asymptotically stable if the inequality (40) along with the condition $\lambda \geq \frac{k^T k}{4}$ is satisfied.

Proof. The proof of this Corollary is similar to the proof of Theorem 2. ∎

Remark 5. It is worth noting that, unlike the approach discussed in [15], our proposed cost-functional takes into account both safety and uncertainty aspects with the inclusion of terms like $\lambda\beta(x)^2$ and $q_{\max}(x)^2$. Furthermore, in comparison to [2], our work addresses uncertainties that are more practical considerations from a system perspective. Additionally, we have introduced a modular design approach, enabling a smooth transition between control laws obtained after solving the optimization problem.

Remark 6. When the uncertainties do not lie within the input range, that is, $f(x) \neq g(x)\phi q(x)$ and $\Delta g(x) \neq g(x)\phi r(x)$, they are classified as unmatched [16]. Accommodating such uncertainties requires modification in the Assumptions 2 and 3. Since unmatched uncertainties cannot be directly compensated through control input, the nominal dynamics (8) is suitably modified by auxiliary dynamics, which is then used to update the safety constraints (9). Consequently, the cost (10) is redefined through appropriate selection of $\lambda\beta(x)^2$ and $q_{\max}(x)^2$, resulting in a modified HJB formulation.

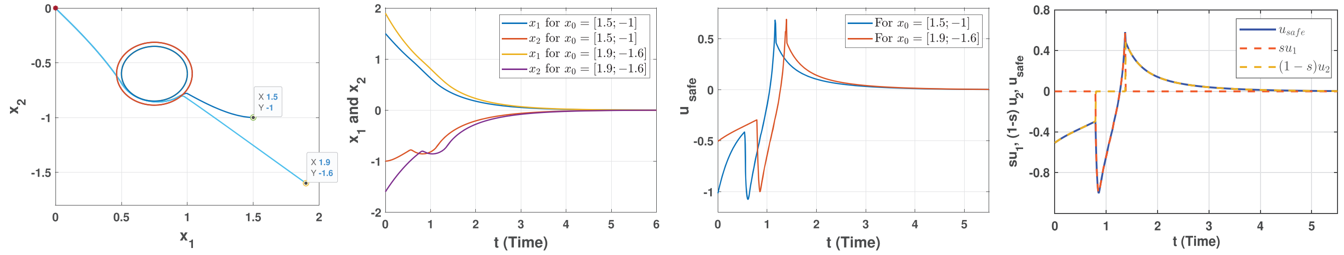
5 | RESULTS AND DISCUSSION

We consider two distinct examples to verify the effectiveness of the proposed safe robust control law for uncertain systems. In the first example, the uncertain system exhibits nonlinearity, while the nominal model is linear in nature. In contrast, in the second example, both the uncertain system and the nominal model are nonlinear. Furthermore, the choice of $q_{\max}(x)^2$, $\lambda\beta(x)^2$, γ, ϵ and ϕ is based on a systematic balance between uncertainty range, control effort, and safety guarantees, while ensuring compliance with Assumptions 1–3 and the hypotheses of Theorem 2.

5.1 | Example 1

Consider a second-order nonlinear system with matched uncertainty, as introduced in [15], described below

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} p_1 x_1 cos(\sqrt{x_2}) + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}0\\ 1\end{bmatrix}p_2 x_2^2 u,$$



(A) Phase Portrait (B) State Trajectories versus time (C) Control Input (D) Switching Function

FIGURE 2 Simulation results for safe robust control for two different sets of initial conditions. The circle in (a) separates the safe and unsafe regions.

where $p_1 \in [-0.2, 2]$ and $p_2 \in [0.1, 0.7]$ are the uncertain parameters. By comparing with (7), we obtain the functions, $f(x) = \begin{bmatrix} x_2 & 0 \end{bmatrix}^T$, $g(x) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $r(x) = p_2 x_2^2$ and $q(x) = p_1 x_1 cos(\sqrt{x_2})$ along with the matrix $\phi = I$. Thus, the state and the input uncertainties can be expressed as, $\Delta f(x) = \begin{bmatrix} 0 & q(x) \end{bmatrix}^T$ and $\Delta g(x) = \begin{bmatrix} 0 & r(x) \end{bmatrix}^T$. The bounds on parameters p_1 and p_2 are chosen to satisfy the requirements specified in Assumptions 2 and 3. With $\epsilon = 0.5$, $q_{\max}$ can be selected as $2\sqrt{2}\ |x_1|$. Now, to transform the robust control problem into an optimal control problem, we consider the nominal system as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

To compute (13), it is necessary to ensure compliance with the safety constraint (18) and determine the switching function (14). Additionally, the solution V_{safe} can be obtained from the HJB Equation (23). The safety parameters to compute (18) are $h'(x) = (x_1 - 0.75)^2 + (x_2 + 0.6)^2 - (0.2598)^2$ with tuning parameter $\alpha = 20$. $h'(x)$ is the over-approximate region of $h(x)$ with $h(x) = (x_1 - 0.75)^2 + (x_2 + 0.6)^2 - (0.25)^2$, which is introduced to handle the uncertainties and guarantee safety effectively. The switching function s in (14) is implemented with the chosen value of $\gamma = 0.1$. To obtain the approximate value of V_{safe}, we employ the successive Galerkin method [6] for the numerical solution of (23) with the value of $\lambda = 16$ and $\beta_2(x)^2 = s(x)^2(8x_2^2 + 9.6x_2 + 2.88)$. More details can be found in the supplementary material. It is important to note that when the safety constraint is inactive, that is, $\hat{C}_s > 0$ and $s = 0$, the HJB Equation (23) simplifies to the HJB Equation (27).

Throughout the simulations, we represent the equilibrium point with "$*$", the initial point with "$\circ$", the inner circle is the obstacle, and the outer circle is $h'(x)$. We first consider two different initial conditions as $[1.9, -1.6]$ and $[1.5, -1]$ with $p_1 = p_2 = 0.2$. Figure 2a,b show the phase portrait and variation of states with time, respectively. It is imperative to note that the trajectory traverses through the over-approximate region but does not enter the unsafe set $h(x)$ as seen from Figure 2a. Figure 2b clearly illustrates the convergence of states. Figure 2c depicts the variation of control input for two different initial conditions, eventually converging to zero. Figure 2d depicts the behavior of the control law based on the shift of the switching function. With the value of $s = 1$, the control law u_1 is active, as the value of the switching function varies from 0 to 1, the control law is updated as a combination of u_1 and u_2 and when $s = 0$, u_2 is used. This behavior is clearly visible in Figure 2d, where the switching function, along with the control inputs are shown for the initial condition $[1.9, -1.6]$. It is crucial to note that the switching function s plays a significant role in defining a transitional interval, allowing the control law to smoothly shift from u_1 to u_2.

Next, to gain some insights into the proposed safe robust control scheme, we choose the initial condition as $[1.9, -1.6]$ and select 20 random values for the parameters $p_1 \in [-0.2, 2]$ and $p_2 \in [0.1, 0.7]$. Figure 3a depicts the trajectories avoiding the unsafe region and reaching the final point despite being subjected to randomly chosen values of uncertainties. For improved clarity, the zoomed view closely displays several trajectories that circumvent the unsafe region even under the influence of randomly chosen parameters in their respective bounded range. The convergence of states and control inputs is illustrated in Figure 3b,c, respectively. The variation of uncertainties consequently leads to the variation in the states, thereby causing variation in $h(x)$ depicted in Figure 3d.

5.2 | Example 2

Consider the following nonlinear system

$$\dot{x}_1 = x_1^2 + x_2^2 - x_1(x_1^2 + x_2^2),$$
$$\dot{x}_2 = -x_1^2 + x_2^2 - x_2(x_1^2 + x_2^2) + u + p x_1 sin\ x_2,$$

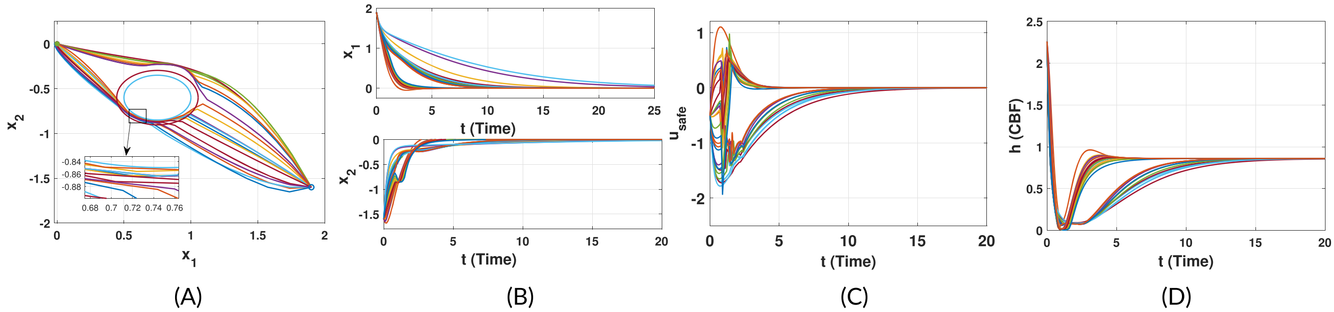


FIGURE 3 Safe robust control design for 20 random values of uncertainties $p_1 \in [-0.2, 2]$ and $p_2 \in [0.1, 0.7]$ with initial condition set as $x_0 = [1.9; -1.6]$ and $\alpha = 20$. (a) Phase Portrait (b) State Trajectories versus time (c) Control Input (d) Variation of CBF.

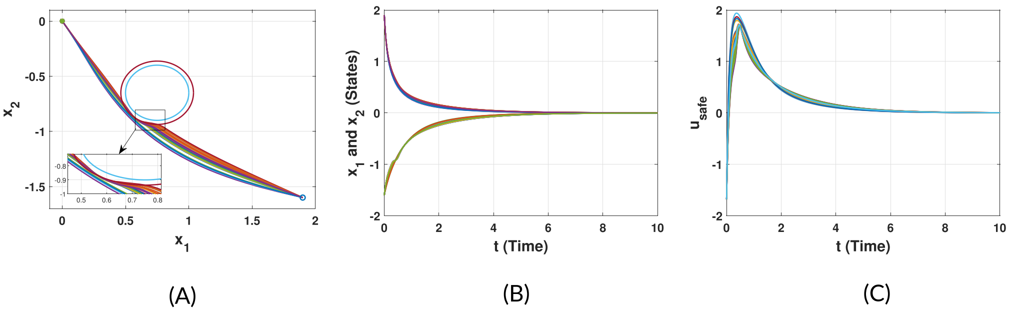


FIGURE 4 Illustrating the efficacy of safe robust control law for the nonlinear nominal system. (a) Phase Portrait of x_1 and x_2 (b) Evolution of state trajectory (c) Evolution of control plot.

where $p \in [-1, 1]$ is the uncertain parameter with the following nominal dynamics

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2^2 - x_1(x_1^2 + x_2^2),\\ \dot{x}_2 &= -x_1^2 + x_2^2 - x_2(x_1^2 + x_2^2) + u.\end{aligned}$$

As the system dynamics involved in the nominal model are nonlinear, a constrained nonlinear safe optimal control problem is solved to obtain a safe robust control law (13).

Following a technique equivalent to the one indicated in Example 1, an analogous study is conducted with initial condition $[1.9, -1.6]$. The simulation is carried out with respect to the nonlinear nominal dynamics, thus modifying the safety constraint and other parameters accordingly. To solve the nonlinear constrained optimal control problem, we modify the successive Galerkin technique by adding additional integral terms (as discussed in supplementary material) and use the successive Galerkin technique to solve the unconstrained nonlinear optimal control problem.

In our simulation experiments, we followed a consistent procedure of extracting 20 random samples from the interval $[-1, 1]$ for the parameter p. These samples were then used to evaluate the performance of our designed controller. Figure 4a–c describe the phase portrait plot, state trajectory plot, and control input for 20 random values of $p \in [-1, 1]$ and provide a clear visualization of the convergence of state trajectories under various values of $p \in [-1, 1]$. Notably, as observed in Figure 4a,b, irrespective of the specific samples drawn from the uncertainty set, all states consistently converge to the equilibrium point and avoid the unsafe region, which confirms the robustness and effectiveness of the proposed control law.

5.3 | Comparative study with existing literature

To compare the proposed scheme with existing results from literature, we consider the system dynamics and parameters as described in Example 1 with the initial condition as $[1.9, -1.6]$ and $p_1 = 0.8$, $p_2 = 0.5$.

Figure 5 presents a comparison of the proposed approach with robust control [15] work and safe optimal control approach [2].

In particular, Figure 5a compares the trajectories of the states x_1 and x_2 obtained from the proposed control law (13) with that presented by Lin [15] as robust control law. An interesting contrast between the two approaches is that the latter primarily addresses the robustness of the system, while the former maintains safety along with robustness. The result for the control action is shown in Figure 5b. It is evident that the designed control law exhibits versatility in achieving the objective of maintaining safety and robustness. Precisely, we can observe that the robust control law coincides with the proposed control law when the safety

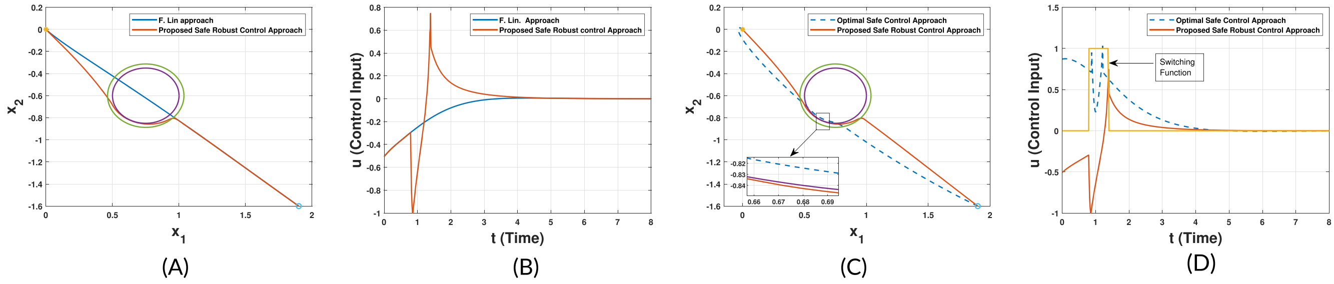


FIGURE 5 (a,b) Comparison plot between F. Lin's approach [15] and proposed safe robust control approach. (c,d) depicts the difference between optimal safe [2] and proposed safe robust control approaches. (a) Phase Portrait of x_1 and x_2 (b) Control Input (c) Phase Portrait of x_1 and x_2 (d) Evolution of control plot.

constraint $\hat{C}_s$ is inactive and a control input u_1 is used when $\hat{C}_s$ is active. Hence, we can conclude that the proposed control law distinctly addresses safety compared to the robust control strategy.

Figure 5c,d are used for the comparative study of the safe robust control law (13) and optimal safe control law [2]. Due to the presence of bounded uncertainties, the behavior of the trajectories changes, as shown in Figure 5c. The trajectories tend to enter the unsafe region, thus violating the safety criterion. However, the proposed controller effectively prevents from entering the unsafe region even in the presence of uncertainty. The simulation presented in Figure 5d compares the ability of the control laws to remain in a safe region. The significant difference between the two approaches lies in the formulation of the objective function, which subsequently impacts the design of the control law. Unlike the optimal safe control law, the proposed control law utilizes the switching function to depict the behavior of the control law for $0 < C_s < \gamma$, which is a combination of u_1 and u_2. In Figure 5d, the switching function visually represents how the proposed control law behaves in response to variations of safety constraints.

6 | CONCLUSION

We presented a safe-Writing -original draft preparation; robust control approach that is solved using an optimal control strategy for nonlinear control systems. Control barrier functions (CBF) are used to map the safety aspect of the system, which is enforced as a constraint along with the aim of minimizing the HJB equations. A safe- robust control law is derived that decouples the safety and robustness based on the safety constraint. We showed that the proposed control design exhibits stability and keeps the system safe. For verification of the proposed technique, we considered nonlinear systems with matched uncertainties and obtained the simulation results using MATLAB.

An important direction for future research is the experimental validation of the proposed approach, along with the incorporation of practical aspects such as actuator saturation, time delays, and sensor noise. Also, the impact of time-varying obstacles could be an important and worthy area for further investigation.

AUTHOR CONTRIBUTIONS

Anjali Tiwari: conceptualization; writing – original draft; investigation; validation; methodology; writing – review and editing. **Niladri Sekhar Tripathy:** conceptualization; validation; writing – original draft preparation; writing – review and editing. **Mohammadreza Chamanbaz:** writing – original draft preparation; validation; writing – review and editing. **Roland Bouffanais:** writing – original draft preparation; validation; writing – review and editing.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

No new data generated.

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SUPPORTING INFORMATION

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APPENDIX A. PROOF OF LEMMA 2

Proof. Let us define a matrix $\mathcal{A} = \frac{1}{2}(\epsilon^{-1}I - I)^{\frac{-1}{2}} g(x)^T \frac{\partial h'(x)}{\partial x}^T sk - (\epsilon^{-1}I - I)^{\frac{1}{2}} \phi q(x)$. We know that $\mathcal{A}^T\mathcal{A} \geq 0$, therefore

$$\begin{aligned}
&\frac{1}{4}\left(g(x)^T \frac{\partial h'(x)}{\partial x}^T sk\right)^T (\epsilon^{-1}I - I)^{-1} g(x)^T \frac{\partial h'(x)}{\partial x}^T sk \\
&\quad - \left(\frac{1}{2} g(x)^T \frac{\partial h'(x)}{\partial x}^T sk\right)^T \phi q(x) \\
&\quad - \frac{1}{2} q(x)^T \phi^T g(x)^T \frac{\partial h'(x)}{\partial x}^T sk \\
&\quad + q(x)^T \phi^T (\epsilon^{-1}I - I)\phi q(x) \geq 0.
\end{aligned}$$

On rearranging above inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(g(x)^T\frac{\partial h'(x)}{\partial x}^T sk\right)^T \phi q(x)\\
&\quad+\frac{1}{2}q(x)^T\phi^T g(x)^T\frac{\partial h'(x)}{\partial x}^T sk+q(x)^T\phi^T\phi q(x)\\
&\quad\leq \epsilon^{-1}q(x)^T\phi^T\phi q(x)+\frac{1}{4}\left(g(x)^T\frac{\partial h'(x)}{\partial x}^T sk\right)^T\\
&\quad(\epsilon^{-1}I-I)^{-1}g(x)^T\frac{\partial h'(x)}{\partial x}^T sk.
\end{aligned}$$

■